TRANSACTIONS OF
THE
OPTICAL SOCIETY

VOL. XXVII. 1925-26 No. 1.

# SOME RECENT IMPROVEMENTS IN MODERN OPHTHALMIC LENSES

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[Communicated by Mr A. H. Emerson.]

*MS. received, 29th October, 1925. Read and discussed, 12th November, 1925.*

*ABSTRACT.* A short account of the ordinary forms of invisible fused bifocals is given together with some of the limitations inherent in them. The reason for the unavoidable increase of the chromatic aberration in the positive combinations is shown to lie in the use of ordinary dense flint to produce the reading addition.

The optical and physical properties necessary in two glasses which shall fuse together to form a bifocal with a minimum of colour are stated, and new bifocals made of glasses with the required properties are described.

Possible modifications in the shape of the reading portion are shown, giving greater control of the prism introduced and permitting distinct distant vision through the bottom of the lenses.

A brief description of some improved forms of medium power and cataract lenses is included.

## INTRODUCTION

THERE is perhaps nothing so remarkable in the ophthalmic world to-day as the rapid increase in the use of invisible bifocal lenses; a description of some recent improvements in the fused varieties may therefore be of general interest to the ophthalmological members of this Society. Before describing these in detail it will be well to look at some of the limitations inherent in the usual forms and, later, to show how these have been met.

## LIMITATIONS OF ORDINARY BIFOCALS

It is common knowledge that when a fused bifocal lens has a large power added for reading it has a considerable amount of chromatic aberration; this is due to the fact that the crown and flint lenses available for fusing are combined in a manner

opposite to that required to produce a positive achromatic lens; the want of achromatism is therefore aggravated instead of being remedied. This colour has confined their use, among critical patients, to low reading additions and very small reading segments.

Another restriction is that there has been a very limited control of the centering of the reading portion. The amount of prism in the vertical meridian at the part of the reading portion most used is dependent on three factors, and these have governed the amount of prism in invisible bifocals whether fused or solid. They are:

(1) the power of the distance portion,
(2) the amount added for reading,
(3) the diameter of the reading circle.

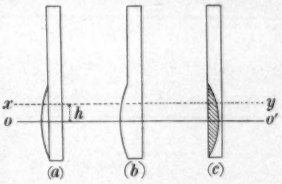


Fig. 1

Fig. 1 shows that there is the same amount of prism in invisible fused, solid, and knife-edge cemented bifocals of similar powers and sizes of reading circle. The flat form has been chosen as being the simplest since only negligible changes in the centering will take place when the meniscus form is used: (*a*) represents a plano distance lens with a knife-edge segment cemented to it, (*b*) is the same in solid form, and (*c*) is a fused lens. All have the same power $D$ and the same size of reading circle. The optical axes of the reading portions lie in the line $oo'$ which passes through the optical and geometrical centres of all of them. But it is a part midway between the centre and the edge of the circle, say a distance $h$ cm. from the optical axis, which is most used, and here there will be an amount of prism, base down, equal to $h \times D$ prism dioptres in each case, as the power of each lens is the same; the fact that the last lens is a combined one does not alter the result as the prismatic effect is dependent on the total power.

We have seen that the reading segment always introduces prism base down; to obtain a lens without prismatic effect at any given point above the centre of the circle we must grind the other side of the main lens convex to introduce prism base up, equal and opposite to that introduced by the reading segment. This of course is assuming always that the distance portion of the lens is correctly centered. It will be seen, therefore, that the centering of ordinary fused bifocals is very little under control as the segments cannot be made very large on account either of the extra colour introduced or of the lens becoming undesirably thick; improvements in both these respects are therefore very desirable.

We have seen that the chromatic aberration in the convex combinations is due to the use of flint and crown in a manner which increases rather than decreases the aberration; a reduction of it is therefore only to be obtained by the use of glasses with very different properties from those at present in use.

It has taken a very long time to procure two glasses with the optical constants necessary to make a fused bifocal with a minimum of colour, but thanks to Messrs Chance Bros. and Co., Ltd., who have made special melts of glass, it has been

possible to obtain two kinds of glass which are equal in expansion, and therefore fusible, and with suitable dispersions.

Let us examine briefly the properties most desirable in the glass to be used for the major lens. It should be hard, white, very transparent, have a very low dispersion and preferably be of the same refractive index as the glass ordinarily used in ophthalmic work. The glass we are using in the improved bifocals has all these qualities in the highest degree, except that the refractive index is rather low. To meet this difficulty the blanks are only supplied in the semi-finished state, all the allowance necessary being made on the segment side, until such time as a glass can be procured of the ordinary refraction.

If we were able to obtain two glasses of the same expansion and with suitable dispersions which would combine together to form an achromatic combination, we should still not be able to make every combination of distance power and reading addition achromatic. The reason is, that as the power of the combination increases from zero, the power of the negative component required to neutralize the colour in the positive component also increases at a rate which depends on the ratio of the relative dispersions of the two glasses. Now the outer surfaces of a fused bifocal are controlled by the power of the distance portion and the contact surface is required to vary the addition. Thus, it is obvious that it is not possible to vary the power of the achromatizing lens so as completely to remove the colour in all cases. The amount of colour present depends on the relation between the powers of the distance and reading portions. We see, therefore, that it is not feasible to produce fused bifocals in which the reading addition and distance power shall be continuously variable and at the same time the reading portion remain truly achromatic.

## Partial Removal of Limitations in Newer Bifocals

The next best approximation to a fused bifocal in which the colour is diminished is one in which the colour is not increased. It may easily be shown that the condition for this is that the $\nu$-values, or relative dispersions of the two glasses, shall be the same, the difference in the refractive indices remaining. This condition is very nearly fulfilled by the glass specially made by Messrs Chance.

The glass used for the major lens is of low refractive index and low dispersion; that used for the reading segment is not flint at all, but a crown glass of high index and abnormally low dispersion, the $\nu$-value being approximately the same as that of the major lens. Bifocals made from these two glasses will therefore only have the same amount of colour in the reading portion as an ordinary single lens of the power of the reading portion made from either glass; this is of course very small.

There are several examples of fused bifocals made from these glasses exhibited this evening and the freedom from colour of lenses with large reading additions will be particularly noticeable.

The second limitation mentioned was that concerning the centering of the reading portion. This has been met in part at the same time as the removal of one of the greatest objections to bifocal lenses, namely, the fact that many users have

complained that they cannot see the ground or stairs distinctly when looking through the lower part of their glasses and that there is a deviation produced in the line of sight.

In a modification of the usual form of fused bifocal, recently patented by Mr F. Bruce Watson, and which we have helped to perfect, the reading portion is of such a shape that the wearer sees distinctly through a lens of the power required for distance when he looks through the bottom of his spectacles.

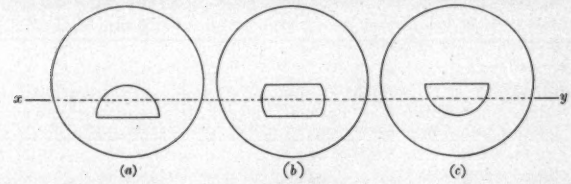



Fig. 2

Fig. 2 shows three of these modifications which, while giving a sufficiently large field for reading, still allow ample room beneath the reading portion for distance. (*a*) has a reading portion which is bounded by a semi-circle and a straight line instead of the usual circle, the rest of the lens being correct for distance vision; (*b*) is a lens with the top and bottom of the reading circle removed, the reading portion thus being nearly rectangular; in lens (*c*) the reading portion is of similar shape to that of (*a*) but inverted.

We saw in Fig. 1 that at the part most used for near work, a few millimetres below the top of the circle, prism base down was always introduced by the reading portion of ordinary fused bifocals. Fig. 3, which contains vertical cross-sections of the lenses in Fig. 2, shows clearly that this prism is now more under control and may be made either base up or down. In (*a*) the prism is base down, the same as in Fig. 1; (*b*) has now the centre of the circle at the part most used and thus produces no prism. In (*c*) the centre of the circle lies above the line of sight and this results in prism base up. Now each of these lenses has the same segment surface and the same reading addition; therefore a different curve will have to be ground on the other side of each if there is to be no prism at the part most used for reading, assuming always that the distance portion is properly centered.

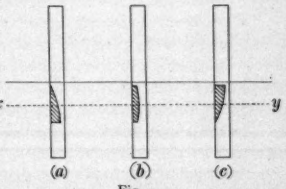



Fig. 3

Hence it will be seen that (*a*) is the most suitable with medium convex powers for distance, (*b*) for very low powers, while (*c*) is the best for concave combinations, as the prism base up due to the reading portion helps to neutralize the prism base down due to the distance portion; in this particular form the optical centres of

both the distance and reading portions can be made to coincide so that there is no jump when passing from one field to the other.

All these shapes of reading segments can be made both in the ordinary fused form and in the improved form which is known as the "AKRO."

## Cataract and Medium Power Lenses of Reduced Weight

Another direction in which work is proceeding is in the improvement of medium power lenses and cataract lenses. A reduction in weight and improvement in appearance together with an increase in the field of view is aimed at, again by the fusion of suitable glasses. The lenses take the form of a segment placed centrally in a larger lens, this central portion only being used.

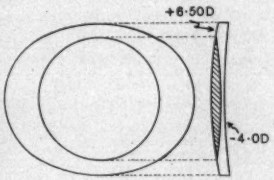
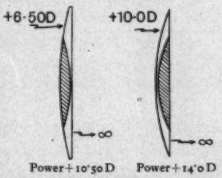


Fig. 4 Power+5·0D

Fig. 4 shows two views of a medium power lens of decreased weight, the power being + 5·0 dioptres, while the external curvature is exactly that of a + 2·50 dioptre lens. Such a lens is quite suitable for a rimless mount and from cosmetic considerations is very desirable.

Fig. 5 shows two forms of cataract lenses made in the same way, and here again the powers are greatly in excess of the external curvatures, so that they are comparatively light in weight. It is not yet certain whether the best form lens of this type is commercially possible nor how it compares in performance and cost of production with the best form of single lens of this power. What is certain, however, is that if any small sacrifice in efficiency be necessary, then it will be made willingly by persons afflicted with eyesight requiring the addition of lenses of this high power, in order to avoid the exceedingly ugly appearance caused by the deep curves necessary in the best form of single lens.



Power+10·50D Power+14·0D
Fig. 5

It was hoped to have the figures available in time to be given at this meeting, but unfortunately this was not possible. Nevertheless, it was felt that a short description of the lenses in their present stage of development was of sufficient interest to merit inclusion.

A list of the fused lenses exhibited is given on the next page.

# LIST OF FUSED LENSES EXHIBITED

| No. | Kind | Distance D | Reading D | Shape of segment | Special features |
|---|---|---|---|---|---|
| 1 | Akro | 6330 | | | Semi-finished meniscus blank 3 D addition |
| 2 | ,, | +4·50 | +8·50 | | 4·0 D reading addition |
| 3 | ,, | −5·50 | −2·50 | | 3·0 D ,, ,, |
| 4 | ,, | +6·50 | +9·50 | | 3·0 D reading addition. Strong convex |
| 5 | ,, | −2·0 | +2·25 | | 4·25 D ,, ,, |
| 6 | ,, | +2·50 | +5·75 | | 3·25 D ,, ,, |
| 7 | ,, | +2·0 | +6·50 | | 4·50 D ,, ,, |
| 8 | ,, | −5·50 | −3·0 | | Concave combination |
| 9 | ,, | −8·0 | −5·0 | | Strong concave combination |
| 10 | ,, | +0·75 | +2·75 | *a* | Distance power available below segment |
| 11 | ,, | +2·0 | +4·0 | | ,, ,, ,, ,, edged oval |
| 12 | ,, | plano | +2·50 | | Reading portion without prism |
| 13, 14 | ,, | +0·75 | +2·75 | *b* | Distance power available below segment |
| 15 | ,, | +1·0 | +2·75 | | ,, ,, ,, ,, ,, |
| 16, 17 | ,, | +1·0 | +3·0 | | ,, ,, ,, ,, ,, |
| 18 | ,, | +1·0 | +3·75 | | ,, ,, ,, ,, ,, |
| 19, 20 | ,, | +1·0 | +3·50 | *c* | Distance and reading centres coincident |
| 21 | ,, | −2·0 | plano | | Reading portion without power or prism |
| 22 | ,, | −3·50 | −1·75 | | Reading portion with minimum prism |
| 23 | ,, | −4·50 | −3·25 | | ,, ,, ,, ,, ,, |
| 24 | Twofo | plano | +1·75 | | Distance power available below segment |
| 25 | ,, | plano | +2·0 | | ,, ,, ,, ,, ,, |
| 26 | ,, | plano | +2·50 | | ,, ,, ,, ,, ,, Crookes A |
| 27 | ,, | +1·0 | +3·0 | | ,, ,, ,, ,, ,, |
| 28, 29 | ,, | +1·0 | +3·50 | *a* | ,, ,, ,, ,, ,, |
| 30 | ,, | +1·25 | +3·0 | | ,, ,, ,, ,, ,, |
| 31 | ,, | +2·0 | +4·50 | | ,, ,, ,, ,, ,, |
| 32 | ,, | +2·0 | +4·75 | | ,, ,, ,, ,, ,, |
| 33 | ,, | −5·0 | −3·0 | | Prism under segment removed |
| 34 | ,, | plano | +2·50 | *b* | Reading portion without prism |
| 35, 36 | ,, | +1·0 | +3·25 | | |
| 37 | ,, | −6·0 | −4·0 | *c* | Distance and reading centres coincident |
| 38 | ,, | −2·50 | plano | | Reading portion without power or prism |
| 39 | Trifo | plano | +2·25 | | Intermediate power +1·25 D |
| 40, 41 | ,, | +0·50 | +3·0 | | ,, ,, +1·50 D |
| 42 | ,, | +1·0 | +3·25 | | ,, ,, +2·0 D |
| 43 | ,, | +2·0 | +4·0 | | ,, ,, +3·0 D |
| 44 | ,, | −1·0 | +1·0 | | ,, ,, 0 D |
| 45 | Light | +5·0 | | | +6·50 −4·0 +7·0 |
| 46 | Weight | +5·50 | | | |
| 47 | Lenses | +10·0 | | | +7·50 ∞ |
| 48 | | +13·0 | | | +9·0 ∞ |
| 49 | | +16·0 ⌒ +3·0c. | | | +10·0 +3·0c. |
| 50 | | +18·0 ⌒ +3·0c. | | | +11·50 +3·0c. |

## DISCUSSION

Dr E. E. Maddox (communicated): It will be interesting to see whether the new shapes will pass the crucial test of public wear. Two thoughts occur to me. (1) The eye does not seem to like angles or corners in the reading segments. It is the nature of the human eye to "wander round" and it loses its bearings if a sector of the pupil is invaded by an angle. (2) Though in reading and writing we preferentially use the upper part of our segments, as Mr Emerson truly notes, do not ladies often use the lower parts of the lens for their knitting and crochet—things which they often do by indirect vision downwards while really looking at things or people higher up? With this exception, however, I think it a very sensible experiment to place the segment well above the lower edge of the major lens, especially if the corners can be rounded.

The "reduced weight" lenses for rimless mounts and cataract lenses strike me as admirable.

The making of fused bifocals is a marvellous art and I would not like to seem to criticize so valuable a piece of work.

Mr H. C. Raxworthy: Some years ago I conceived the idea of making a fused bifocal with a circular sphere—prism segment of a prismatic power sufficient to place the optical centre of the reading portion in the correct position. I found, however, that in those instances where such correction is most needed, i.e. in deep concave lenses, the main lens would have to be too thick for practical wear. We must therefore look upon the new forms as described by Mr Emerson as only partially solving our ever-pressing problem of fused bifocal segment centration.

The extension of the distance power to an area below the segment is very useful, especially in the higher reading additions, to enable wearers to walk downstairs or to look with comfort over the heads of a crowd.

The new lenses are certainly ingenious and I must congratulate Mr Bruce Watson on his invention, and also the United Kingdom Optical Co. for their clever productions. What I miss, however, is a technical description of the methods used in the manufacture, but I assume they are a trade secret too good for publication.

Mr W. M. Brett: Bifocal spectacle lenses, of the fused variety with improved chromatic correction, are much to be desired, and the forms described by Mr Emerson appear to offer this advantage.

The reading segment placed above the lower edge of the lens, allowing the wearer to obtain distinct vision in the field immediately before his steps, is an advantage that will be very desirable especially for advanced presbyopes; the confusion caused by the immediate field of vision becoming blurred by the reading segment is a serious handicap to the constant use of bifocals, and adds to the risks of accident to the wearer. It would seem that the new form introduced by the

United Kingdom Optical Co. would overcome this objection, and will probably prove very successful in general practice.

It is a matter of congratulation that we have had presented to us a paper dealing with an advance in ophthalmic lens construction of British design and manufacture.

Mr F. Bruce Watson: I should like to state publicly my appreciation of the work of Mr Emerson, Mr Culver, and others of the United Kingdom Optical Co. for the splendid lens they have produced from my original idea. I expected a practical lens from my necessarily rough description, but the lens which they have produced is, in my opinion, a triumph for British optical lens work.

Mr H. H. Emsley: I am sure we are all grateful to Mr Emerson for describing, and bringing for us to see, such excellent examples of fused bifocal work. There seems to me to be room for an authoritative paper dealing with the underlying principles of bifocal lenses and, in the case of desiderata that clash, advancing reasons for the adoption of one as against another. For example, we can either "centre" the near portion, in which case there will be a jump in passing from this to the distance portion, or we can eliminate the jump by leaving the near portion decentered.

Again there are certain practices, such as making the optical centre of the near portion $\frac{1}{4}$ inch or 6·25 mm. below that of the distance portion, that appear to have nothing to recommend them except the fact that they have existed for so long. Certain terms too are used in a misleading manner and these should be defined.

Mr Emerson: I think that Dr Maddox will find that the new forms will pass the crucial test fairly easily. A practical trial of all three shapes was carried out in the cemented form before the fused variety was produced. No discomfort was felt, and the habit of the eye of "wandering round" was not interfered with as this usually occurs with the eye in the relaxed condition when looking into the distance over the top of the reading portion.

I agree that there are special cases in which near vision is required when using the bottom of the lens. This fact, however, may be taken to be the exception that proves the rule.

There is certainly room for a paper such as Mr Emsley describes. From some quarters we hear of nothing but segment centration; in others maximum optical performance is demanded without regard to cosmetic effect.

# THE RELATION OF VISUAL ACUITY AND ACCOMMODATION TO AMETROPIA

By W. SWAINE, B.Sc.

*MS. received, 29th October, 1925. Read and discussed, 12th November, 1925.*

*ABSTRACT.* The author has previously published a theoretical table showing how degrees of ametropia affect visual badness (i.e. inverse of acuity). This paper is concerned with the comparison of artificially produced results on his own eye, of statistical figures, etc., with the theoretical table.

An attempt is made at the same time to isolate the portion of the astigmatic pencil used in letter vision in astigmatic defects, with and against the rule, by using apparent accommodative esophoria as indicator.

Published work by the author in the same connexion, and hitherto not indexed, is also collected.

## Snellen Letter Acuity

The term visual acuity is usually restricted to imply acuteness of vision as measured by the ability to recognize standardized letters. In view of other acuity tests, e.g. gratings, illiterate E, etc., which, unfortunately, do not give directly comparable values, it is advisable to specify more particularly the type of visual acuity under consideration. The term *Snellen Letter Acuity* describes implicitly the character of the visual acuity here under consideration.

### *Visual Badness*

For purposes of description, and even during practical work, the writer has found it more convenient to use an inverse scale describing the visual "badness" or visual "blurring" ($B$). For most defects of eyesight, visual badness increases with the degree of error, and one ventures to suggest that the idea may have advantages during clinical routine. Again, some writers distinguish between "vision" or $V$, which is the acuteness of vision before correction, and "visual acuity" or $V.A.$ after correction. This paper is mainly concerned with the relation between ametropia and "vision," the latter being expressed in terms of visual badness.

If $d$ represents the distance at which letters are placed, and $D$ represents the distance at which the letters so defined must be placed in order to subtend vertically a height of 5 minutes at the eye, then

$$V = \frac{d}{D},$$

whilst

$$B = \frac{D}{d},$$

where $B$ is the visual badness as determined by Snellen test type.

There is no need to review the historical development of the measurement of acuity as this is now readily accessible in Helmholtz' book (1) and has been extended by H. L. in the American translation (2). One may, however, supplement these by reference (*a*) to Donders (3), who gave more particulars concerning the early history of letter tests, (*b*) to the second German edition of Helmholtz' book (4) for illumination factors, and (*c*) to Tscherning (5), Nutting (6), Sheard (7), and Hartridge (8). All these are mainly concerned with maximal acuity of vision. Although a knowledge of the dependence of visual badness on kinds and degrees of ametropia is of some considerable clinical value, no attempt has been made, so far as the writer is aware, to place the relationship on a more exact basis. Its study has been left to individual experience.

At various ophthalmological congresses (9) objections have been raised against Snellen measurements, partly because unit acuity is not the average highest acuity, partly because the gradation of type is not convenient, and partly because letter vision involves other factors than mere resolving power. The only objection of serious practical importance, in the writer's opinion, lies in the variable ease with which letters can be recognized. If Snellen letter acuity of 1·3 is, on the average, found to be normal (10), this fact does not militate against the use of Snellen letters for the measurement of visual acuity; it affects only the design of the chart in so far as finer gradations are required.

Table I. *Letter Acuity*

| 1 | American | | Swaine | |
|---|---|---|---|---|
| | 2 | 3 | 4 | 5 |
| A | — | — | 1·28 | 1·1 |
| B | 1·0 | 1·0 | 1·17 | 1·0 |
| C | ·79 | 1·27 | 1·12 | ·96 |
| D | ·82 | 1·22 | 1·13 | ·97 |
| E | ·85 | 1·18 | 1·15 | ·98 |
| F | ·81 | 1·23 | 1·16 | ·99 |
| G | ·92 | 1·09 | 1·01 | ·86 |
| H | ·92 | 1·09 | 1·21 | 1·04 |
| K | — | — | 1·16 | ·99 |
| L | ·71 | 1·40 | 1·44 | 1·23 |
| N | ·85 | 1·18 | 1·23 | 1·05 |
| O | ·80 | 1·25 | 1·21 | 1·04 |
| P | ·81 | 1·23 | 1·10 | ·94 |
| R | ·88 | 1·14 | 1·08 | ·92 |
| S | ·89 | 1·12 | 1·17 | 1·00 |
| T | ·74 | 1·35 | 1·25 | 1·07 |
| U | ·79 | 1·27 | 1·30 | 1·11 |
| V | ·78 | 1·28 | 1·36 | 1·16 |
| X | — | — | 1·21 | 1·04 |
| Y | ·81 | 1·23 | 1·15 | ·98 |
| Z | ·84 | 1·19 | ·96 | ·82 |
| | | Mean 1·13 | | |

2. "Difficulty" scale. 3. Acuity. Letter B = 1.
4. Actual acuity. 5. Acuity. Letter B = 1.

The variable ease with which letters may be recognized is a factor which affects the accuracy of measurement for all acuities, low or high. If certain astig-

matic defects render the recognition of one letter easier, and another more difficult, Snellen measurements tend to lose accuracy unless some control be present. Possibly letter vision is the best measure obtainable at the present time in such circumstances. Other factors being equal, the ability to recognize different letters varies with the simplicity of outline, and Table I shows the results of the American Medical Association (11) compared with some obtained by myself on my right eye (12). Column 2 gives a figure relative to B as unity, which expresses the relative acuity should the same sized Snellen letter mentioned in Column 1 be just recognized. This relative figure will be the inverse of the apparent acuity of a person tested if his Snellen letter B acuity be unity. Thus the Snellen letter acuity as determined in the American investigation and recalculated to reciprocal values by myself is given in Column 3. In Columns 4 and 5 are given my own actual acuities and acuities relative to B as unity respectively. The figures in my own case were obtained when the pupil diameter was 4·5 mm., black letters on a white ground illuminated by electric light, approximately 100 metre-candles, being used.

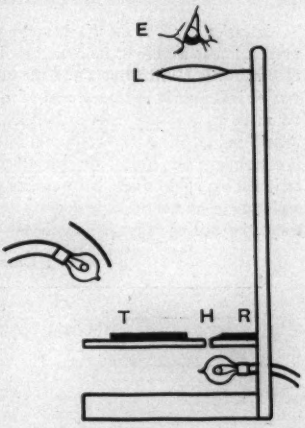


Fig. 1

Evidently letter vision is the recognition of an outline so that many factors are in operation of which a very influential factor is the degree of blurring. The circle of confusion which just obliterates a letter L will decidedly obliterate a letter B. The writer has found the ratio of the diameter of a blur circle to the height of a Snellen letter at the point of obliteration. The results are given in Table II. The ratio varies with different letters but not in proportion to the acuity. On the other hand the ratio does not differ greatly from 1·92, the mean value (13). The writer has used as an approximate ratio 2 to 1, i.e. if a blur circle exceeds half the vertical height, the letter is just obliterated.

Table II. *Blur Circle Ratios*

| Letter | B | C | D | E | F | N | O | P | R | S | Z | Mean |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Ratio | 2·1 | 1·85 | 1·75 | 2·10 | 2·10 | 2·05 | 1·37 | 2·0 | 2·1 | 1·9 | 1·8 | 1·92 |
| V.A. | 1·17 | 1·12 | 1·13 | 1·15 | 1·16 | 1·23 | 1·21 | 1·10 | 1·08 | 1·17 | ·96 | |

These ratios were obtained in the following way. A + 6·0 dioptre lens *L* (Fig. 1) was placed before the right eye *E*, which viewed illuminated Snellen letters *T*. The eye and lens were moved away from the chart and when the letter just faded from recognition the projected diameter of the blur circle was measured

with the left eye. To render the circle more easily detectable, a small hole *H* was punctured through the cardboard and illuminated from behind, the millimetre scale *R* being placed close beside it. The blur was then compared with the vertical height of the letter.

## Classification of Ametropia

In order to cover the whole range of degrees and kinds of ametropia, the writer prefers to use a table of spheres combined with cylinders which correct the various errors. Table III illustrates how the older classification is related to it (14). The table can be indefinitely extended when required to include still higher degrees and can be made to suit any finer subdivision. All varieties of regular astigmatism are considered as being corrected with minus cylindrical lenses. In this form the table is convenient for clinical interpretation, as the most common method of measuring astigmatism subjectively is by means of minus cylinders. The table thus presents a picture of all the possible degrees and kinds of ametropia, natural or artificial, as they arise during subjective routines.

Table III. *Classification of Ametropia*

| | Sph. | | | | | | Cyl. |
|---|---|---|---|---|---|---|---|
| −4 −3 −2 −1 | 0 | +1 | +2 | +3 | +4 | +5 | |
| Myopia | Em. | Hypermetropia ← ← ← | | | | | 0 |
| Compound myopic astigmatism | Simple myopic ast. | Simp | | | ← ← ← ← Compound | | −1 |
| | | | le h | | ← ← ← hyperopic | | −2 |
| | | Mixed astigmatism | | ype r | ← ← ast. | | −3 |
| | | | | | opic ← | | −4 |
| | | | | | | ast. | −5 |

The arrows at the right-hand top corner suggest the direction in which presbyopia begins to complicate the issue with the introduction of "absolute" ametropic defects. We are here mainly concerned with manifest ametropia.

## Manifest Ametropia and Visual Badness

### *Statistical Table*

In a preliminary examination of the records of ophthalmic opticians from all parts of the country, the writer has jotted down the visual badness for chosen defects for males and females, below 35 years, whose corrected acuity was not less than $\frac{6}{9}$. The results of this somewhat cursory examination are given in Table IV. The figures given are averages and the relative weights are given in brackets. A statistical analysis of acuity will fail to give information of much value unless pupil diameter, illumination, etc., are stipulated, and unless there is uniformity in the method of inducing a person to exert his utmost to recognize a letter or vice versa. A striking feature in the original records was the different acuities recorded in equal degrees of the same defect.

Table IV. *Manifest Ametropia and Visual Badness* (*Statistical*)

Random selection from Opticians' Records.
Age 10–35. Corrected Acuity $< \frac{6}{9}$. Majority—cyl. ax. H.
Charts, pupil diameter, and illumination not known.

| Sph. | | | | | | | | | Cyl. |
|---|---|---|---|---|---|---|---|---|---|
| −4 | −3 | −2 | −1 | 0 | +1 | +2 | +3 | +4 | |
| (4) | (3) | (2) | (3) | (5) | (20) | (8) | (1) | | |
| 10·3 | 8·7 | 6·0 | 2·5 | ·9 | ·89 | 1·1 | ·9 | — | 0 |
| (1) | (4) | (2) | (5) | (4) | (8) | (6) | | | |
| 10 | 7 | 8 | 4·5 | 2·0 | 1·7 | 2·7 | — | — | −1 |
| | | | (1) | (1) | (8) | (3) | (3) | (1) | |
| — | — | — | 6 | 8 | 3·5 | 3·2 | 3·7 | 3 | −2 |
| | (1) | | | (2) | | (3) | (5) | | |
| — | 12 | — | — | 9 | — | 5·3 | 5·2 | — | −3 |
| | | | (1) | (1) | (1) | | | | |
| — | — | — | 12 | 12 | 8 | — | — | — | −4 |

### *Theoretical or Schematic Table*

As a working guide to students, the writer has compiled and published a theoretical or schematic table (15), indicating the relation of visual badness to manifest ametropia for an average size of pupil.

Table V. *Manifest Ametropia and Visual Badness* (*Theoretical*)

Age 10–35. Pupil diameter, 3 mm.

| Sph. | | | | | | | | Cyl. |
|---|---|---|---|---|---|---|---|---|
| −3 | −2 | −1 | 0 | +1 | +2 | +3 | +4 | |
| 10 | 6 | 2 | ·5 | ·5 | ·5 | ·5 | ·5 | 0 |
| 12 | 8 | 4 | 2 | 1·5 | 1·5 | 1·5 | 1·5 | −1 |
| 14 | 10 | 6 | 4 | 4 | 3 | 3 | 3 | −2 |
| — | 12 | 8 | 6 | 6 | 5 | 5 | 5 | −3 |
| — | 14 | 10 | 8 | 8 | 8 | 8 | 7 | −4 |

This theoretical or schematic table, of which Table V is an extract, was based partly on leading experiments, partly on experience, and partly in accordance with certain easily memorized rules, which the writer has derived from time to time and published with certain reservations.

The rules referred to are as follows:

(1) As regards distance acuity an hypermetropic eye with sufficient accommodation is comparable with an emmetropic eye.

(2) Assuming an average pupil diameter, and that a letter is obliterated when each blur circle is greater than one-half the vertical height of the letter, then "each quarter of a dioptre of myopia increases the visual badness by unity."

(3) Assuming that an astigmatic eye can accommodate for the circle of least confusion, and that it does so in letter vision, then, since the diameter of the circle of least confusion is, theoretically, one-half the diameter of the blur circle in a corresponding degree of myopia, "each half dioptre of hyperopic, simple or compound, astigmatism increases the visual badness by unity."

(4) By experiment the writer found that in simple myopic astigmatism with minus cylinder horizontal, "approximately each quarter dioptre of defect increases the visual badness by unity." When the axis of the correcting cylinder is vertical, half a dioptre of astigmatism was required. These rules lose accuracy in fractional units of badness, i.e. in high degrees of visual acuity.

The schematic table was based on data obtained at different times before the writer conceived the idea of a systematic tabulation. For the purposes of this paper the writer decided to analyse the accuracy of the assumptions upon which the schematic table was partly based. He had misgivings about the assumption that an eye with hyperopic astigmatism focusses at the circle of least confusion, despite the fair agreement of the schematic table with practical experience. He made a series of more systematic observations and tabulated simultaneously

(*a*) visual badness for different degrees and varieties of ametropia artificially produced on his own eye and

(*b*) accommodative esophoria measured with the Maddox groove whilst the eyes fixed the distance test types.

In artificially produced astigmatic defects, observations were made in four cardinal orientations of the cylinder axis, and Table VI shows the relation of visual badness to each degree and variety of artificial ametropia. The pupil diameter was 4 mm.

The number of prism dioptres of accommodative esophoria associated with each degree and variety of artificial ametropia is shown in Table VII.

The results shown in Tables VI and VII must not be interpreted as the visual badness or accommodative esophoria produced by the lenses quoted, but as that produced by the artificial defect created. In other words the lenses quoted are equal and opposite to those used to create the defect.

*Comparing Tables IV (Statistical) and V (Theoretical)* we notice that the visual badness figure in Table IV is slightly better than that in Table V when the visual badness exceeds 5 units; it is slightly worse for higher acuities.

Table VI. *Manifest Ametropia and Visual Badness (Artificial—Swaine, R. E.)*

Test Chart. Union Reverted.
Illumination, 100 m.c. Pupil diameter, 4 mm. Distance of test, 6 m.

| Sph. | | | | | | | | Cyl. | |
|---|---|---|---|---|---|---|---|---|---|
| −3 | −2 | −1 | 0 | +1 | +2 | +3 | +4 | D | Ax. |
| 10 | 6 | 3 | 1·0 | ·9 | ·8 | ·8 | ·8 | 0 | |
| — | 12? | 5 | 2·5 | 1·25 | 1·25 | 1·25 | 1·25 | | H. |
| 12 | 8 | 4 | 1·5 | 1 | 1 | 1 | 1 | | V. |
| 10 | 10 | 6 | 2 | 1·5 | 1·5 | 1·5 | 1·5 | −1 | 45° |
| 11 | 10 | 5 | 2·5 | 1·5 | 1·5 | 1·5 | 1·5 | | 135° |
| 10·3 | 10 | 5·2 | 2·1 | 1·4 | 1·4 | 1·4 | 1·4 | | Mean |
| — | 14 | 6 | 4·5 | 3·5 | 3·5 | 2·5 | 2 | | H. |
| — | 12 | 5 | 3 | 1·75 | 1·5 | 1 | 1 | | V. |
| — | 12 | 10 | 6 | 4·5 | 3 | 3 | 3 | −2 | 45° |
| — | 18 | 10 | 6 | 4 | 3 | 2·5 | 2·5 | | 135° |
| — | 14 | 7·7 | 4·9 | 3·4 | 2·7 | 2·2 | 2·1 | | Mean |
| — | — | 14 | 8 | 8 | 8 | 6 | 5 | | H. |
| — | — | 10 | 5 | 3·5 | 3 | 2·5 | 2·5 | | V. |
| — | — | 13 | 12 | 10 | 10 | 6 | 6 | −3 | 45° |
| — | — | 13 | 12 | 8 | 4 | 4 | 4 | | 135° |
| — | — | 12 | 9·2 | 7·3 | 6·2 | 4·6 | 4·3 | | Mean |
| — | — | — | 12 | 10 | 8 | 6 | 6 | | H. |
| — | — | — | 6 | 5 | 3·5 | 3·75 | 3·5 | | V. |
| — | — | — | 12 | 8 | 7 | 7 | 5 | −4 | 45° |
| — | — | — | 12 | 12 | 12 | 7 | 5·3 | | 135° |
| — | — | — | 10·2 | 8·7 | 7·6 | 5·9 | 5·0 | | Mean |

Table VII. *Manifest Ametropia and Accommodative Esophoria in Δ's.*

Test Chart. Union Reverted.
Illumination, 100 m.c. Pupil diameter, 4 mm. Distance of test, 6 m.

| Sph. | | | | | | | | Cyl. | |
|---|---|---|---|---|---|---|---|---|---|
| −3 | −2 | −1 | 0 | +1 | +2 | +3 | +4 | D | Ax. |
| — | 1·5 | 1·0 | 1·1 | 2·0 | 6·0 | 11·0 | 14·0 | 0 | |
| | | | 1 | 2·0 | 6·0 | 11·0 | 13·0 | | H. |
| | | | 1 | 1·7 | 2·5 | 7·0 | 10·0 | | V. |
| | | | 1 | 1·5 | 6·0 | 10·0 | 13·0 | −1 | 45° |
| | | | 1 | 2·0 | 6·0 | 10·0 | 13·0 | | 135° |
| | | | 1 | 1·1 | 6·0 | 11·0 | 14·0 | | H. |
| | | | 1 | 2·0 | 4·0 | 5·0 | 8·0 | | V. |
| | | | 1 | 1·5 | 6·0 | 8·0 | 13·0 | −2 | 45° |
| | | | 1 | 1·1 | 6·0 | 10·0 | 13·0 | | 135° |
| | | | 1 | 1·0 | 5·0 | 10·0 | 13·0 | | H. |
| | | | 1 | 2·0 | 2·5 | 5·0 | 8·0 | −3 | V. |
| | | | 1 | 1·0 | 7·0 | 9·5 | 12·0 | | H. |
| | | | 1 | 2·0 | 5·0 | 6·0 | 7·0 | −4 | V. |

*Comparing Tables IV (Statistical) and VI (Artificial)* we notice that again there is passable agreement in the lower degrees of visual badness, but no consistent agreement for the higher degrees. This may be due to the few letters and coarse gradations of the testing charts. For stricter comparison between these two tables, the top figure of each rectangle in Table VI should be taken, as the majority of the results in Table IV were for defects corrected by minus cylinders with axes horizontal.

The results shown in Tables IV and VI indicate that there is a progressive decrease in visual badness from myopic astigmatisms to the hyperopic astigmatisms, and, in this respect, these tables differ from the theoretical Table V which was constructed by fitting.

## Accommodative Esophoria

Although this title suggests relationship with muscle balance, the essential idea of this series of observations (Table VII) was to determine what portion of the astigmatic pencil the astigmatic eye focusses. The figures showing the apparent esophoria on the hypermetropic side of the table are merely indicators of the accommodation in play. When the row corresponding with zero cylinder is considered we may assume that the apparent esophoria measures the accommodation

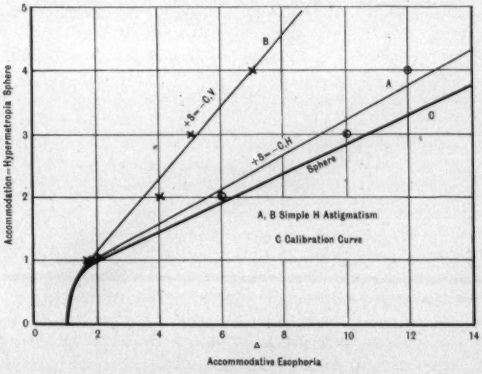



Fig. 2

as expressed in the degree of hypermetropia (Fig. 2). If these be plotted—simple hypermetropia against esophoria—we have an accommodation calibration curve for considering what accommodation is exerted in astigmatic vision. This, coupled with mental observations of the blurred appearance of the letter when the artificial defects were produced, leads to results which may have an important bearing on other aspects of refraction work besides its bearing on visual acuity. Table VIII

shows the accommodations exerted as determined by the apparent esophoria and its calibration curve (Fig. 2). As regards the absolute measure of the accommodation figure (Table VIII) objections may be raised, as possibly there is lag or over-accommodation in the apparent hypermetropia corresponding with the different apparent esophorias. But the differential effects are, I think, beyond question.

Table VIII. *Manifest Ametropia and Accommodation. From Table VII using calibration curve*

| Sph. | | | | | Cyl. | |
|---|---|---|---|---|---|---|
| 0 | +1 | +2 | +3 | +4 | D | Ax. |
| 0 | 1 | 1·9 | 3·0 | 3·75 | 0 | |
| 0 | 1 | 1·9 | 3·0 | 3·5 | −1 | H. |
| 0 | 1·0 | 1·2 | 2·1 | 2·8 | | V. |
| 0 | 0 | 1·9 | 3·0 | 3·75 | −2 | H. |
| 0 | 1 | 1·6 | 1·7 | 2·4 | | V. |
| 0 | 0 | 1·7 | 2·8 | 3·5 | −3 | H. |
| 0 | 1 | 1·2 | 1·7 | 2·4 | | V. |
| 0 | 0 | 2·1 | 2·7 | 3·3 | −4 | H. |
| 0 | 1 | 1·7 | 1·9 | 2·1 | | V. |

The nest of diagrams (Fig. 3) serves to show graphically the portion of the pencil which the astigmatic eye chooses. From observation it was found that best vision resulted when the astigmatic eye accommodated for the circle of least confusion, but that this position was most unstable, the tendency being for the eye to accommodate so that a focal line, especially a vertical focal line, was placed at the retina. From the diagrams we observe that there is a smoother curve when the vertical focal line is the more hyperopic with respect to the retina. This indicates that there is much greater precision in accommodation when the axis of the correcting minus cylinder is horizontal, this precision of accommodation being even antagonistic to the interests of best vision. When the defect is such that the correcting minus cylinder is vertical, there is a greater degree of uncertain focussing but better vision is ultimately obtained. There is always a consistent tendency for the eye to focus for the most hyperopic meridian, that is, to accommodate through both the anterior focal line and circle of least confusion. Fig. 4 shows the astigmatic pencil in the case of astigmatism with and against the rule*.

* This figure has been included at the suggestion of Mr C. L. Redding, who kindly read this paper at the Optical Society meeting.

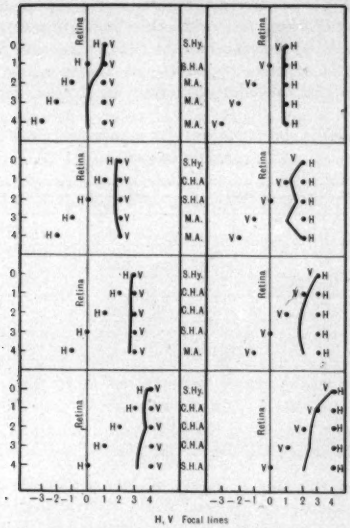



Fig. 3

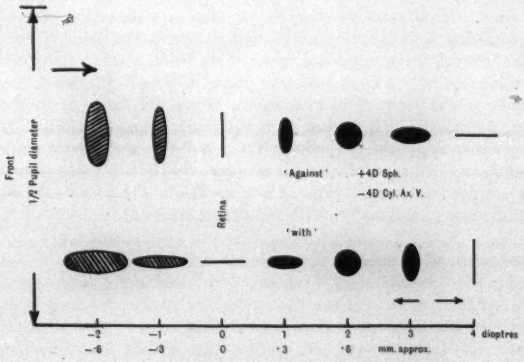



Fig. 4. Simple Hyperopic Astigmatism (Blur sections)

### Artificial Ametropia and Visual Badness (Small Pupil Diameter)

The last series (Table VI), which showed artificial manifest ametropia and visual badness when the pupil was 4 mm., was repeated with a pinhole disk, 1·5 mm. in diameter (Table IX). Such a pupil diameter is extreme and indeed constitutes a limiting diameter at which diffraction effects become noticeable.

Table IX. *Manifest Ametropia and Visual Badness (Artificial)*

Compare with Table VI.
Pupil diameter, 1·5 mm.

| Sph. | | | | | | | | | Cyl. | |
|---|---|---|---|---|---|---|---|---|---|---|
| −4 | −3 | −2 | −1 | 0 | +1 | +2 | +3 | +4 | D | Ax. |
| 6 | 4 | 2 | 1·1 | ·9 | ·7 | ·7 | ·7 | ·7 | 0 | |
| 6 | 5 | 3 | 1·3 | 1·0 | ·7 | ·7 | ·7 | ·7 | −1 | H. |
| 6 | 5 | 3·5 | 1·2 | 1·0 | ·7 | ·7 | ·7 | ·7 | | V. |
| 7 | 6 | 3·5 | 1·5 | 1·5 | 1·0 | ·9 | ·9 | ·9 | −2 | H. |
| 8 | 6 | 4 | 2 | 1·5 | 1·0 | ·9 | ·9 | ·9 | | V. |
| 8 | 6 | 4 | 3 | 2·0 | 1·5 | 1·5 | 1·5 | 1·2 | −3 | H. |
| 8 | 6 | 4 | 3 | 1·5 | 1·5 | 1·0 | 1·0 | 1·0 | | V. |
| — | 8 | 5 | 3 | 2·5 | 1·7 | 1·6 | 1·5 | 1·5 | −4 | H. |
| — | 6 | 4 | 3 | 2·5 | 1·6 | 1·5 | 1·2 | 1·0 | | V. |
| — | — | — | — | 5 | 4 | 3·5 | 3·5 | 3·5 | −8 | H. |
| — | — | — | — | 5 | 4 | 3·0 | 3·0 | 2·5 | | V. |

### Myopia and Visual Badness

Empirical rules have been given by the late Dr Ettles (16) and by Mr Laurance (17), whilst the writer has given a theoretical table (18), which latter takes into account pupil diameter. These are placed in Table X in juxtaposition with results given in the tables already given above.

Table X. *Myopia and Visual Badness*

| Rate per $B=1$ | Myopia −4 | −3 | −2 | −1 | Pupil diameter | Author | Ref. | Source |
|---|---|---|---|---|---|---|---|---|
| ·4 | 10 | 6 | 3 | 2 | Average | Ettles | (16) | ? Empiric |
| ·25 | — | — | 6 | 3 | ,, | Laurance | (17) | ? |
| ·3 | — | 10 | 6 | 2 | ,, | Swaine | Table V | Theoretical |
| ·3 | — | 10 | 6 | 3 | 4 mm. | ,, | ,, VI | Artificial |
| ·67 | 6 | 4 | 2 | 1·1 | 1·5 mm. | ,, | ,, IX | ,, |
| ·4 | 10·3 | 8·7 | 6 | 2·5 | Average | ,, | ,, IV | Statistical |

If we assume that a letter is blurred beyond recognition when the blur circle

exceeds one-half the vertical height of a letter, then it may be shown by elementary considerations that

$B = 1{\cdot}33\ p.R,$

where

$B$ = visual badness = reciprocal visual acuity,

$p$ = pupil diameter in mm.,

$R$ = refractive error referred to the principal point of the eye; e.g. degree of Myopia.

*Proof*

Angle subtended by Snellen letter $= {\cdot}5B$ mins.

Angle subtended by blurring circle $= \frac{{\cdot}5B}{1{\cdot}92}$ mins. $= \frac{B}{1{\cdot}92} \times \frac{300}{206000}$.

Let $b$ = diameter of blur in mm. and assume that the pupil lies in the principal plane. Then

$$\frac{p}{b} = \frac{\frac{1}{F}}{\frac{1}{F-R} - \frac{1}{F}} = \frac{F-R}{R}.$$

But $\frac{b}{1000} \times (F-R)$ = angle subtended by blurring circle $= \frac{pR}{1000} = \frac{300B}{1{\cdot}92 \times 206000}$,

$$\therefore B = 1{\cdot}33pR.$$

When $p = 3$ mm., $B = 4R$ and when $p = 6$ mm., $B = 8R$.

Diffraction will modify this relation when $p$ falls below 1·5 mm.(13). In consulting rooms $p$ varies between 3 and 6 mm., whence $B$ varies between $4R$ and $8R$, or, expressed conversely, the rate of change of myopia will vary between a quarter and one-eighth dioptre per unit badness. It was not considered worth while to make any more elaborate calculations on the theoretical side(19), as, for example, assuming pupillary position in front of the principal plane of the eye, or to enter into a discussion on aberrational effects(20) which undoubtedly enter into the question. Despite the disagreement of the theory with the statistical results of Dr Ettles and those from opticians' records, the writer feels that on the average of his own experience the relations given suffice for most practical purposes.

## Astigmatism and Visual Badness

Here again, although the eye does not accommodate for the circle of least confusion, serious error is not involved on the average as the astigmatic eye tends to be in a state of fluctuation, its vision fluctuating slightly and frequently passing through this circle. As Gullstrand(20) points out, aberrational effects ought not to be ignored, but in the coarser degrees of acuity arising from deeper astigmatic errors, it is more than likely that serious haphazard irregularities of curvature are present modifying the conclusions of a more exact investigation. From the statistical table it will be found in general that vision is slightly worse on the average than that indicated by the elementary theory.

### More Generalized "Refraction" Equation

In the text of this paper refraction errors are described in terms of the correcting lenses, sphere and negative cylinder.

Let

$F$ = reduced power of eye,
$X$ = reduced convergence for retina,
$S$ = power of correcting sphere,
$C$ = power of correcting cylinder,
$A$ = accommodation in action,
$a$ = distance from axis of point of intersection of a ray with the retina, the ray striking the edge of a pupil of diameter $p$ and refracted by the defective eye as determined by $F, X, S, C, A$ above.

As above

$$\frac{p}{a} = \frac{\frac{1}{F-S+C+A}}{\frac{1}{F-S+C+A} - \frac{1}{X}} = \frac{X}{X-F+S-C-A}.$$

For a meridian at right angles, i.e. along the axis of $C$, in which $C = 0$, we have

$$\frac{p}{b}(\text{say}) = \frac{X}{X-F+S-A}.$$

These two equations will give details of the elliptical, circular, or linear blurs due to astigmatism for any eye in any refractive condition, restricted of course to paraxial assumptions and regularity of defect.

When $A = 0$, for distance correction,

$$\frac{p}{a} = \frac{X}{X-F+S-C} \quad \text{and} \quad \frac{p}{b} = \frac{X}{X-F+S},$$

and, if we assume for the sake of argument $X = F = 60$, then

$$\frac{p}{a} = \frac{60}{S-C} \quad \text{and} \quad \frac{p}{b} = \frac{60}{S},$$

i.e.
$$a = \frac{p(S-C)}{60} \quad \text{and} \quad b = \frac{p.S}{60}.$$

When
$$C = 0, \quad a = b = \frac{pS}{60},$$

which is the equation already derived above. At the circle of least confusion

$$a = -b,$$

whence
$$2S = C,$$

i.e.
$$a = -b = \frac{-p}{60} \times \frac{C}{2},$$

i.e. the diameter of circle of least confusion in a specified degree of astigmatism is one-half the corresponding blur circle in a similar degree of myopia.

## Depth of Focus

All subjective tests involving the use of letters will show a range of focussing as determined by the blur necessary to obliterate a letter. It is usual to apply the Hooke-Helmholtz figure of 1 minute of angle as expressing resolving power in this connexion.

From the relation

$$\theta = pA,$$

where $\theta$ is the angular diameter of a blur on the retina, we have

$$\theta = 1 \text{ minute} = \frac{1}{3438},$$

so that $A$, which measures the excess or deficiency of refractive power about the conjugate focus, is given by

$$pA = \frac{1}{3438},$$

or

$$A = \text{depth of focus} = \frac{\cdot 291}{p \text{ mm.}}.$$

If

$$p = 3 \text{ mm.}, A = \cdot 097 = \cdot 1 \text{ nearly.}$$

This is the same figure obtained by Southall (19) by more elaborate calculation. This figure is applied to express the possible error in measurement of ametropia using the Snellen chart. It is assumed that a person with Snellen acuity of unity, tested at 6 m., may be over-corrected with plus spheres by $\frac{1}{6}$ dioptre, the artificial far point, plus the depth of focus figure 0·1, i.e. nearly a quarter dioptre.

The writer looks at this from a different angle. If the person has a Snellen acuity of 1, then his resolving power may be expressed in angular measure as one-half the height of the letter, i.e. $2\frac{1}{2}$ minutes. His depth of focus figure is therefore increased in this proportion, i.e. $2\frac{1}{2}$ times the Hooke-Helmholtz result, i.e. the depth of focus is ·25 dioptre. Similarly, if the maximum acuity of any corrected ametrope is a visual badness $B$, then the depth of focus becomes $\cdot 25B$. It is the writer's contention that the Hooke-Helmholtz resolution of 1 minute is an extreme figure which corresponds with a Snellen letter acuity of about $2\frac{1}{2}$ (6/2·4). If the average Snellen letter acuity be 1·5 ($\frac{6}{4}$), then the writer would prefer the figure $1\frac{1}{2}$ minutes or 100 sec. as expressing the average resolving power, giving as a value for the depth of focus $\cdot 25B = \cdot 24 \times \frac{4}{6} = \frac{1}{6}$D.

## Accommodation and Depth of Focus

Accommodation is usually measured by reading fine types described by Jaeger or Snellen. These types are graded to suit the acuity of the eye for the various distances. Generally this grading is insufficient, and, in fact, near work testing is generally in a very unsatisfactory state. There is no need why this should be, since the depth of focus is the same at distance and near for the same pupillary diameter. If the Snellen chart be continued in the form convenient in near vision, calibrated

in units of badness for corresponding near points—this can evidently be accomplished in one single near vision chart provided with a sliding scale of badness with automatic distance indicator—the error due to depth of focus at the distance test is of the same magnitude and in like direction at near, so that the accommodation may be directly measured. The near type read must correspond with the highest acuity obtained in the distance test.

During the tests on artificially produced astigmatism it was observed that the accommodation exerted when hyperopic astigmatism was present was somewhat relaxed if larger type were read than corresponded with the smaller type just recognized. As reading matter is usually several times larger than the smallest recognizable, we are led to the conclusion that the uncorrected astigmat does not accommodate for his most hyperopic meridian when reading, but rather tends to accommodate in these circumstances for the most myopic meridian.

## Subjective Testing

When the physiologically active eye is being tested subjectively, the usual aim is to neutralize the most hyperopic or least myopic meridian first, and then to neutralize the astigmatism by equalizing the appearance of radiating lines using concave cylindrical lenses. Having obtained the badness of vision of the unaided eye, various methods are adopted to reach this result. One method consists in adding plus spherical lenses unless vision begins to blur so that accommodation for letter vision is neutralized. The spherical lens measures what the writer has called Snellen letter hypermetropia. Similarly Snellen letter myopia is first neutralized with the weakest minus sphere which fails to improve vision. It is a well-known fact that in most cases artificial mixed astigmatism is then produced, indicating that accommodation has been exerted in letter vision so that the retina lies within the interval of Sturm. The above tables can be used to interpret more precisely these steps. As we pass from left to right across the tables the badness improves more rapidly within the compound myopic astigmatism region, slackens within the mixed astigmatism, and then very slowly improves within the compound hyperopic astigmatism region. The essential aim of the step is to bring the eye to the artificial condition of mixed astigmatism. One rule we may observe in this connexion: if vision (badness) fails to improve by unity for an addition of $-0{\cdot}5$ sphere we have reached the limit of minus spheres. Vice versa, when $+0{\cdot}5$ sphere begins to decrease vision (increase badness) by unity, we have reached the limit of acceptance of plus spheres.

## Other Comparative Acuity Tests

As results expressing the acuity of the writer's right eye for photometric balancing and aligning power have already appeared in these *Transactions* (21), references are given at end. His grating acuity has also been published, but further details are here given of some observations taken three years ago.

Gratings consisting of black lines on a white ground were approached from a

point beyond the limit of resolution. At a certain point (*a*) the lines were uncertainly distinguished, the distance being recorded. When the point of certain perception was reached the distance was again recorded (*b*). Within this point the Morocco leather effect (*c*) reached a maximum (*d*), when it gave place to the Helmholtz wave (*e*) (23). Finally a definite point was reached at which the lines appeared straight (*f*). In Table XI the grating subtenses are in seconds and include the sum of a dark and white space, for which constant subtense, when the dark and white were varied relatively, the phenomena appear to occur simultaneously.

Table XI. *Grating Results*

Daylight. Pupil diameter, 2·5 mm.

| Grating width | | | Appearance | Subtense sec. |
|---|---|---|---|---|
| Black mm. | White mm. | Interval mm. | | |
| 2 | 4 | 6 | a | 109 |
| 5 | 1 | | b | 120 |
| 4 | 2 | | c | 140 |
| | | | d | 171 |
| | | | e | 330 |
| | | | f | 1250 |
| 2 | 3 | 5 | a | 106 |
| 4 | 1 | | b | 133 |
| 3 | 2 | | c | 153 |
| | | | d | 230 |
| | | | e | — |
| | | | f | 1380 |
| 1 | 3 | 4 | a | 114 |
| 2 | 2 | | b | 147 |
| 3 | 1 | | c | 171 |
| | | | d | 250 |
| | | | e | — |
| | | | f | 830 |

## REFERENCES


(1) Helmholtz. *Physiologische Optik*, 1st edn.
(2) —— *Physiological Optics*, English Translation, vol. 2, "H.L."
(3) Donders. *Anomalies of Refraction and Accommodation* (1864) 194.
(4) Helmholtz. *Physiologische Optik*, 2nd edn.
(5) Tscherning. *Physiologic Optics*, English Translation.
(6) Nutting. *Applied Optics*.
(7) Sheard. *Physiological Optics*.
(8) Hartridge. "On the Physiological Limits to the Accuracy of Visual Observations, etc.," *Phil. Mag.* 46 (July, 1923) 49.
(9) See Ref. (11); also Reports by G. Lindsay Johnson, *Brit. Opt. Journ.* (1904, 1905); also Henker's *Introduction to Theory of Spectacle Lenses*, English Translation, 192–6.
(10) Guild, J. *Dictionary of Applied Physics*, vol. 4, p. 73.

(11) Sheard. *Amer. Journ. Physiol. Opt.* 2 (1921) 168.
(12) Swaine. "Optical Initiate," *Optician* (13th April, 1923) 156; also Ref. (13).
(13) —— *Ibid.* (10th August, 1923) 447.
(14) —— *Ibid.* (25th September, 1925) 95.
(15) —— *Ibid.* (2nd October, 1925) 110.
(16) Ettles. *Myopia.* Lecture to the Institute of Ophthalmic Opticians (1912)
(17) Laurance. *Visual Optics*, chapter on Myopia.
(18) Swaine. *Optician* (22nd June, 1923) 348.
(19) Southall. *Amer. Journ. Physiol. Opt.* 1 (Oct., 1920) 277.
(20) Gullstrand. Ref. (2), vol. 1, p. 372.
(21) French, J. W. "The Unaided Eye," *Trans. Opt. Soc.* 20 (1918–19) 209; 21 (1919–20) 1, 127.
(22) Swaine. *Optician* (9th October, 1925) 138 and Ref. (13).
(23) Helmholtz. Ref. (2), 35.

## DISCUSSION

Mr T. Smith: Mr Swaine has given us an account of a series of observations of undoubted importance, and has made comparisons with corresponding quantities derived theoretically. In his theoretical method he appears to have followed Helmholtz in taking blur circles on the retina as determined by rays as a basis. From some of the figures given it is apparent that this theory is not reasonably valid over part of the field investigated, and it would be of value to have the variations in path length at the retina expressed in wave-lengths added to the particulars already given. It should be realized that the value of the paper in no way depends upon the adoption of one theory rather than another, and when printed it will no doubt receive the close attention it evidently calls for.

Mr E. F. Fincham: The unsuitability of the Roman alphabet as a basis of a test for visual acuity, even when distorted as in the Snellen test types, is apparent to all who practise sight-testing. It is extremely difficult to determine the size of the smallest letter which the subject is able to read, because of the great differences of legibility of letters of equal size. This fact prevents the examiner from gaining reliable information regarding the degree of the subject's refractive error directly from the recorded acuity.

The attempts to introduce more suitable forms of test objects do not appear to have met with much success, and experienced workers with the Snellen letters find that they are able to acquire a more or less rule of thumb method of interpreting the results which they obtain, although the difficulty of making an exact record of the acuity still remains.

In his work Mr Swaine has, by means of many experiments, found the true value of each of the letters which are in general use, and has given in his paper complete tables which will allow the user of the Snellen types to make a reasonably accurate and direct estimation of the degree of ametropia from the record of the

acuity. Mr Swaine's results will, I think, be of great assistance to all who are engaged in subjective testing for refractive errors.

The author's method of determining the amount of accommodation in use by measuring the degree of accommodative esophoria would prove extremely valuable in many visual experiments. I have, however, some doubt as to the reliability of a method which determines the amount of accommodation by the measurement of another function, which, although associated, is not inseparably linked to it. As Mr Swaine has been able to make a calibration curve showing the relation of the two functions, he has evidently found the method reliable in his own case.

Prof. A. F. C. Pollard (communicated): The interesting papers by Mr Swaine and Mr Emsley recall to mind some investigations upon mental habit, conducted at the Leland Stanford Junior University, California, which are of the utmost importance to those who study methods of measurement in which the personal judgment is a predominating feature, such as the present-day subjective methods of testing optical imperfections of the eye.

It is known that acquired associations sometimes determine preferences for letters and it is probable that the results of a Snellen chart test may be largely due to mental habit and give a wrong deduction as to imperfections of vision.

Letters are extremely complex forms with many mental associations differing in different people and the influence of form on the "badness of vision" indicated by Mr Swaine's patient work may be largely due to mental habit.

It would appear that the angle which two distant white spots on a dark ground (or two black spots on a white ground) subtend at the eye when distinctly resolvable would be a more reliable test for the distinctness of vision and less liable to associated judgments than the estimation of the distinctness of a letter. The spots could be made to approach or recede from one another and the line joining them at right angles to the visual axis could be rotated about it. The results could then be plotted and an attempt made to correct the eye for normal vision of the spots.

I am aware that the test by Thompson's ametrometer is similar to the one suggested but I have never seen it used.

In the light of the investigations upon mental habit referred to above, it is quite hopeless to expect anyone to divide correctly a space by inspection, though in accordance with the expectations of probability some persons may do so at certain times. This is of importance to instrument design and suggests that the use of the vernier is imperative. It is of course quite impossible to discuss here the highly important work which has been done upon mental habit, but those who desire their methods of measurement to be as scientific as present-day knowledge of the "personal equation" will allow should consult the work entitled *Experiments in Psychical Research at Leland Stanford Junior University* by J. E. Coover.

Mr Swaine: Mr Smith has kindly reminded me that the wave theory cannot be ignored when elaborating a theory to cover the present investigation. I had tacitly assumed that a theory on a ray basis would have been ample to explain the grosser

degrees of visual badness under consideration. It will certainly be of value to re-examine the question from the wave theory point of view.

In reply to Mr Fincham the accommodation calibration curve using spherical lenses was tested immediately before and after the astigmatic vision measures with good agreement. Care must be exercised that the attention is fixed on the finest Snellen type recognizable. There is then good agreement from day to day.

The remarks of Prof. Pollard, calling attention to work by psychologists, leads one to philosophical trains of thought. There is need of a greater interchange of ideas between workers in physics, physiology, pathology, and psychology in the case of a science like ophthalmology and it is to be hoped that the ophthalmological branch of this Society may become a means of ensuring this.

# IRREGULAR ASTIGMATISM OF THE EYE: EFFECT OF CORRECTING LENSES

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*MS. received, 5th November, 1925. Read and discussed, 12th November, 1925.*

*ABSTRACT.* Experiments to test the grating acuity of the eye in different meridians reveal the prevalence of irregular refraction, or astigmatism as it is usually called. Comparisons are made between the ordinary subjective findings and tests made on gratings. The effects of changes of lens power on the acuity in different meridians of the eye are considered. A redistribution of the acuity over the several meridians during accommodation is indicated.

## I. Introductory

At each stage of the process concerned with supplying a spectacle lens for an ametropic eye, a more or less unavoidable experimental error is liable to creep in, even if precise methods are used. The methods at present generally in use as routine procedure are subject, in addition, to what may be classed as avoidable errors due to the somewhat lower standard of precision in practice and the more or less rough approximations made. The powers of the two or more separated lenses in the trial frame—the spheres being symmetrical (equi-bi) and the cylinders plano—are added together and the result incorporated in a single thin lens, which may be of an appreciably bent form, meniscus or toric. No allowance is made for the separations of the trial lenses nor of the change of vertex power introduced by the simple transposition to some other form. The possible cumulative effects of these inaccuracies and the difficulties to which they give rise in the processes concerned with supplying the lens have been well described before this Society on a previous occasion (1, 2).

In practical ophthalmic work, at any rate on the manufacturing side, there is beginning a definite trend in the direction of the elimination, or at least diminution, of these errors, as exemplified by the tentative efforts towards specifying spectacle lenses according to their back vertex powers and by allowing for the change in the latter, when lenses are transposed from one form to another, by grinding the lens surfaces to previously-calculated curves. Moreover, attention is being directed towards the use of lenses of forms calculated to reduce the oblique aberration, chiefly astigmatism, introduced when the eye, in moving around its centre of rotation, looks through the peripheral portions of the lens.

The question arises, however, as to what order of accuracy is to be aimed at.

The ultimate judge is the eye; and this, being a living organ and therefore conditioned by its interconnexion with the nervous and other systems of the body, is not susceptible to mathematical treatment in the precise manner applicable to other optical systems. It seemed desirable to attempt to obtain experimental data indicating the order of the lens effects that are appreciated by the eye, both when normal and in various sub-normal states, and, if possible, monocularly and binocularly. There would be no point in designing and providing lenses to a limit of accuracy not appreciated by the eye, or in attempting to satisfy acutenesses of perception of the eye that are variable or unstable or local as to area of aperture, and hence not capable of being permanently satisfied by lenses.

## II. The Apparatus

It was necessary to use a method of testing that is positive, sensitive, and capable of comparison with the usual Snellen test chart and, if possible, with the kinds of tests the eye is called upon to face in its everyday work.

In spite of the existence of the Ives Visual Acuity Test Object, the rotating perforated disk suggested by Ferree and Rand (3), the method proposed by these authors depending on the measurement of the eye's speed of discrimination (4), and other tests, it was finally decided to adopt a suggestion made by E. F. Fincham some years ago. His suggested form of comparison grating was made up by Mr C. H. Walters, a research student in the department; a description of the method of making it and an account of the preliminary tests will, it is hoped, be given by Mr Walters on another occasion. He has unfortunately been prevented from completing his description in time for this meeting.

Briefly, the apparatus consists of three circular plane glass disks, about 7 cm. in diameter, set at the three corners of an equilateral triangle. On two of these disks are carefully-made black and white photographic gratings, the two disks being capable of rotation through 180°. Each grating is illuminated from behind through a disk of frosted glass and a second disk of opal glass. The third disk has no grating, but is illuminated through an opal disk and a disk containing a grating exactly similar to the two described. Thus to an observer in front of the apparatus, and too far away to resolve the two gratings, the three disks appear equally bright and exactly similar. The apparatus is arranged so that the gratings and plane disk may be interchanged and the gratings set at any desired inclinations, unknown to the observer.

The whole apparatus is mounted in a blackened wooden box and this, together with the three 40-watt gas-filled opal lamps used for illumination, is mounted on a simple form of carriage. The observer sits facing the apparatus and can pull it towards or away from himself by strings attached to the carriage. From a distance too great for him to resolve the gratings, he pulls the apparatus towards him until he can discriminate either of the gratings. He is called upon to state which of the three disks he refers to and the direction in which the grating lines run. The

distance between the observer and the grating and the direction of the lines are recorded. The pupil diameter is measured at each series of readings.

The intensity of illumination of the gratings was carefully determined photometrically and found to be 4·8 foot-candles; that is, allowing for the fact that the black lines and spaces are of equal width, equivalent to an illumination of 9·6 foot-candles over the spaces. The width of a grating element, including a line and a space, is 3·5 mm. The angle subtended at a distance $d$ metres is therefore $\frac{12}{d}$ minutes. At 6 metres the angle subtended by a whole element is 2 minutes; the angle subtended by a line or space alone is 1 minute, which is the basis of construction of the ordinary Snellen test chart. The letter or Snellen Visual Acuity corresponding to a distance $d$ metres is therefore given by $6 \div \frac{36}{d}$ or $\frac{d}{6}$; alternatively, the denominator of the Snellen acuity fraction can be found by multiplying the angle resolved (expressed in minutes) by 3. This corresponding Snellen acuity is included only for rough comparison purposes; the two acuities, grating and Snellen, are not so simply interrelated.

### III. Moderately High Regular Astigmatism

After some preliminary tests, a case (S. W.—right eye) was investigated in which the usual subjective test had resulted in the prescription:

$$+ 1{\cdot}00 \text{ D.S.} / - 3{\cdot}50 \text{ D.C. ax. H.}$$

With this correction the acuity on the Snellen chart was $\frac{6}{6}$. Careful readings on the keratometer revealed a fairly regular astigmatism as shown in Table I, which

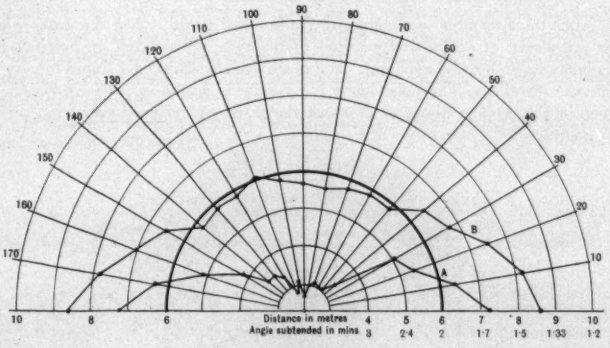
Table I. *Case S.W. Keratometer Readings*

| Meridians | Correction indicated |
|---|---|
| 0°– 90° | − 3·50 D.C. ax. H. |
| 30°–120° | − 2·12 D.C. ax. 30° |
| 45°–135° | − 0·10 D.C. ax. $\begin{cases} 45° \\ 135° \end{cases}$ |
| 60°–150° | − 1·95 D.C. ax. 150° |

gives the results of several readings in each pair of meridians specified. This is a case of astigmatism "with the rule," due to the cornea, which apparently has a regular toric form. Readings were taken on the gratings, first with a + 1·00 D sphere before the eye (to diminish accommodation effects) and secondly with the full correction. The figures are given in Columns 2 and 3 of Table IV (see pp. 36, 37) and polar curves plotted from these figures are shown in Fig. 1.

In the first column of the table the directions 10° to 180° give the meridians of the eye concerned, expressed in the usual standard (or T.A.B.O.) notation; in each case the grating direction observed was perpendicular to the direction specified. In each double vertical column the first set of figures gives the distance $d$ in metres at which the grating was resolved and its direction named, and is therefore proportional to the grating acuity of the eye; the second set of figures gives the angle

δ in minutes subtended at that distance by a whole grating element (line plus space); this figure is proportional, therefore, to what, following Swaine, may be termed the "grating badness." The curves are plotted from the first-named figures; the distance from the origin of any point on a curve represents the distance at which the grating was resolved and is therefore proportional to the grating acuity of the eye along that meridian. The figure at the foot of each vertical column opposite the heading "average angle subtended" is simply the arithmetic mean of the eighteen figures above it and represents therefore the average angle resolved for all meridians. The figure for the Snellen acuity is calculated from this, as explained above.




A with + 1·00 D.S.
B „ + 1·00 D.S./− 3·50 D.C. ax. H.

Fig. 1. Case S.W.

Examination of the results for this case shows that the average value of the angle resolved has been reduced to 1·89 minutes, corresponding to a Snellen acuity of 6/5·7, which agrees fairly well with the finding of the subjective test.

A noteworthy feature is the markedly higher acuity for vertical lines (horizontal eye meridian) even when the correction is being worn. The correction has certainly tended towards equalizing the acuity in the several meridians, as would be expected, but has by no means succeeded in doing so. The observer can still see vertical lines at a distance of more than 8 metres, but is compelled to approach to a distance of less than 6 metres to resolve horizontal lines. Ideally the polar curve after correction would be circular with the origin as centre. This marked preference for lines in a certain direction, after the optical defect of the eye has been fully corrected, constitutes a kind of residual astigmatism, it being understood that this term is not meant to imply that the beam of light emerging from the eye's optical system is astigmatic in the ordinary sense. It may be, in some cases, that this residual non-equality of acuity is not due to an optical defect but that

the reason for it is to be sought in the lens substance or humours or at the retina or even farther back along the optic nerve. The eye may have been uncorrected for years and horizontal lines therefore never distinctly seen. When, after correction, these lines are imaged distinctly on the retina, there is a lack of ability to respond to the stimulus; this is understandable but the precise reason for it is not at the moment clear.

## IV. Observations with Author's Left Eye

### (1) *Subjective Tests*

After the apparatus was made Mr Walters had tested only a few cases when the prevalence of optical irregularities in the eyes, noted often by previous investigators (5, 6), became apparent. In a given eye the acuity was found to vary in different meridians. The polar curve for a quite typical case, *after* correction, is given in Fig. 2. The distances at which the grating is resolved vary from 5·8 to 7·4 metres.

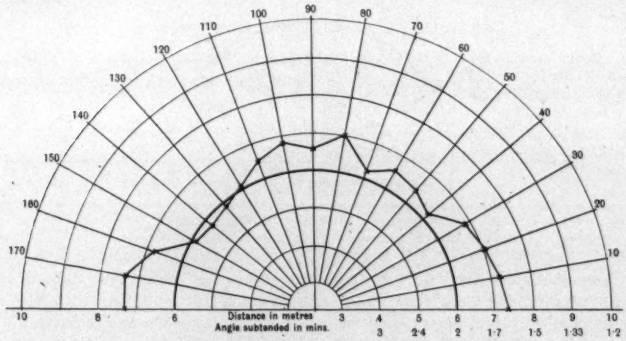



Fig. 2. Irregular Refraction. Typical case—after correction

It was long ago pointed out by Tscherning that failure in obtaining normal visual acuity by the use of the ordinary spherical and cylindrical lenses of the trial case could be traced in many cases, not to a defect of the retina (amblyopia) to which it is usually attributed, but to irregularities in the optical construction of the eye, usually but perhaps not wisely called irregular astigmatism. Tscherning went so far as to suggest that practitioners attempt to correct eye defects with too great an accuracy. He says (5), "it even seems to me that we bid fair to overstep the limit in this sense that we can perceive a tendency to desire to determine these defects too exactly. Quarters of a dioptre are, indeed, superfluous for our test cases and even half dioptres are only rarely indispensable, except for very weak ametropias."

Dr Edward Jackson has also drawn attention to the obstacles these irregularities present to the subjective testing of refraction and the importance of an understanding of them in more cases than is commonly supposed. It is certainly possible that the difficulty experienced with many eyes in trying to determine accurately the amount of astigmatism and especially its direction is caused by the existence of the irregularities referred to. At the same time the important fact must be borne in mind that there are authentic cases where severe headaches have been cured by the prescribing of a cylinder lens of power less than one-quarter of a dioptre.

Since this difficulty of determining astigmatism had arisen with my own eyes, especially the left, it appeared that it would be of interest to investigate the acuity of the left eye in all meridians. This line of attack was not anticipated as a part of the original investigation, but it clearly obtrudes itself, at the outset, as requiring attention. Time has not permitted of an exhaustive enquiry, but the following account may be useful.

The results of tests carried out clinically on my left eye during the past eight years are given in Table II.

Table II

Author's Left Eye. Subjective Tests.

| Date of test | Vision before correction | Correction prescribed | Vision after correction | Remarks |
|---|---|---|---|---|
| 1. Sept. 1917 | — | +0·50 D.C. ax. H. | $\frac{6}{6}$ | Pupil dilated; test by an oculist |
| 2. Aug. 1919 | — | −0·75 D.S. | $\frac{6}{6}$ | Usual subjective test |
| 3. Oct. 1922 | $\frac{6}{18}$ | −1·00 D.S./−0·25 D.C.ax.105° | $\frac{6}{5·4}$ | Usual subjective test |
| 4. May, 1925 | $\frac{6}{18}$ | −1·00 D.S./−0·50 D.C.ax. 130° | $\frac{6}{5·4}$ | Very careful test using Maddox arrow test |
| 5. May, 1925 | $\frac{6}{21}$ | −1·00 D.S./−0·37 D.C.ax. 100° or ax. 125° | $\frac{6}{5}$ | Pupil dilated; 2 drops of H. and C. used. Very careful test |

In test No. 2 the difficulty of establishing any advantage from a cylindrical lens prompted me, in consultation with the examiner, to ignore it. Tests Nos. 3, 4 and 5 have been carried out by my colleague, Mr Redding. Particular care was taken with Nos. 4 and 5. The room was dark, pupil diameter being about 7 mm. In testing for astigmatism, the Maddox arrow test was used in addition to the usual fan and block. I could not observe the regular effect of mutually perpendicular lines being respectively sharp and blurred. In some directions lines were slightly clearer than in others, but they appeared at irregular intervals around from 0° to 180°. I was further confused at recurrent intervals by the well-known wavy appearances of the lines and by their intensely vivid red and green coloration. During all this time of searching for astigmatism I was wearing a −0·75 D sphere.

Eventually it was decided that a cylinder was required with its axis at 100° or 125°, the latter appearing rather the better direction. A − 0·25 D cyl. under-corrected and a − 0·50 D over-corrected. It thus appeared that with a − 1·00 D sphere in position my eye was myopic along the 10° to 35° meridian to the extent of 0·37 D. Supplying a − 0·37 D cyl. axis 100° or 125° resulted in a Snellen acuity of $\frac{6}{6}$ and a fair equality of sharpness of the radiating lines of the chart.

(2) *Keratometer Test*

The keratometer, set at 125°, indicated about − 0·12 D cyl. axis 125°. The amount of astigmatism is too slight for this instrument to provide reliable information as to irregularity.

(3) *Appearance of Illuminated Hole*

A small illuminated hole 0·5 mm. in diameter, observed at a distance greater than a metre, presents to this eye the appearance illustrated in Fig. 3. I have not had the time to investigate this, but its configuration should be noted.

Fig. 3

(4) *Observations on Gratings*

A series of tests was carried out on the same eye, using the comparison grating in the manner described above. The gratings were manipulated by a second operator; I, the observer, had no knowledge as to the directions in which they were being set. The right eye was occluded in every test. A peculiar effect was soon noticed. Frequently, at a given distance, the lines of a grating would become momentarily perfectly clear and then immediately vanish; this occurs when the eye is fogged by plus lenses and is not an effect of accommodation. A determined effort of concentration would often fail to make them clear again at that distance and it would be necessary to bring the grating closer to the eye, sometimes by as much as 75 cm., before the lines were seen with certainty.

A series of readings was therefore taken on lines in two given directions unknown to me, in order to determine the consistency with which settings are made, a knowledge of which will be of service in attempting to analyse the later readings. The results of this test are given in Table III.

Table III

Author's Left Eye—Unaided. Consistency Test.
(Distances in cm.)

| | Direction of eye meridian 160° | | | | Direction of eye meridian 110° | | | |
|---|---|---|---|---|---|---|---|---|
| | Grating first recognized | | Grating resolved with certainty | | Grating first recognized | | Grating resolved with certainty | |
| | Distance | Difference from mean | Distance | Difference from mean | Distance | Difference from mean | Distance | Difference from mean |
| | 610 | 14 | 594 | 54 | 578 | 25 | 550 | 32 |
| | 545 | 51 | 520 | 20 | 580 | 27 | 530 | 12 |
| | 590 | 6 | 525 | 15 | 554 | 1 | 510 | 8 |
| | 600 | 4 | 540 | 0 | 520 | 33 | 510 | 8 |
| | 600 | 4 | 520 | 20 | 540 | 13 | 506 | 12 |
| | 620 | 24 | 540 | 0 | 540 | 13 | 504 | 14 |
| | 610 | 14 | 542 | 2 | 560 | 7 | 518 | 0 |
| Mean | 596 | — | 540 | — | 553 | — | 518 | — |
| Maximum spread | 75 | — | 74 | — | 60 | — | 46 | — |
| Mean diff. from mean | — | 16·5 | — | 16 | — | 17 | — | 12·5 |

I have not previously come across any mention of this effect in connexion with observations on gratings. It seems to be of a similar nature to the "glimpsing" noted by expert microscopists.

Forewarned by this effect, care was taken in subsequent readings to ensure as far as possible that the distance recorded was that at which the lines were seen with certainty; it must be admitted, however, that there is always an element of doubt. The results of several series of readings, the eye being unaided or supplied with lenses as tabulated, are given in Table IV.

## V. Analysis of Results. Author's Left Eye

### (1) *Average Grating Acuity*

Averaged over all meridians, the grating acuity of my left eye when fully corrected is 1·7 minutes or 102 seconds. Expressed as a Snellen fraction this is equivalent to approximately $\frac{6}{5}$, which agrees with the ordinary subjective sight-test on this eye. This is near the usual acuity. A young student tested was able to attain 60 seconds or Snellen $\frac{6}{8}$ and E. F. Fincham approaches 70 seconds, but the ordinarily quoted Helmholtz figure of 60 seconds is not often attained. It will be observed from Columns 11 and 12 that increasing the negative power, and thus forcing the eye to accommodate, produces no change in the average acuity, but there is another effect to which reference will be made below.

## Table IV. *Readings on Gratings*

*d* is expressed in metres and δ in minutes.

| 1 | 2 | | 3 | | 4 | | 5 | | 6 | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Observer | S. W. | | S. W. | | H. H. E. | | H. H. E. | | H. H. E. | | |
| Lens used | +1·00 D.S. | | +1·00 D.S. / −3·50 D.C. axis 180° | | Unaided eye | | Unaided eye | | Unaided eye | | |
| Diameter of pupil | 7 mm. | | 7 mm. | | 6 mm. | | 4 mm. | | 1·75 mm. | | |
| Meridian of eye | *d* | δ | *d* | δ | *d* | δ | *d* | δ | *d* | δ | |
| 10 | 6·33 | 1·90 | 8·18 | 1·47 | 4·50 | 2·67 | 5·35 | 2·24 | — | — | — |
| 20 | 5·40 | 2·22 | 7·50 | 1·60 | 4·25 | 2·82 | 4·40 | 2·74 | — | — | — |
| 30 | 5·05 | 2·38 | 6·75 | 1·77 | 4·25 | 2·82 | 4·40 | 2·71 | — | — | — |
| 40 | 3·32 | 3·62 | 6·46 | 1·86 | 4·00 | 3·00 | 4·15 | 2·90 | 4·50 | 2·67 | Nasal |
| 50 | 3·06 | 3·92 | 5·85 | 2·05 | 3·75 | 3·20 | 4·35 | 2·75 | 5·05 | 2·28 | Above |
| 60 | 3·05 | 3·92 | 5·85 | 2·05 | 4·00 | 3·00 | 4·50 | 2·66 | — | — | — |
| 70 | 3·11 | 3·86 | 5·75 | 2·09 | 4·00 | 3·00 | 5·35 | 2·24 | 5·80 | 2·06 | Templ. |
| 80 | 3·00 | 4·00 | 5·60 | 2·14 | 5·00 | 2·40 | 5·15 | 2·32 | — | — | — |
| 90 | 2·52 | 4·75 | 5·70 | 2·10 | 4·50 | 2·67 | 4·50 | 2·67 | — | — | — |
| 100 | 3·11 | 3·86 | 5·85 | 2·05 | 4·75 | 2·53 | 4·65 | 2·59 | — | — | — |
| 110 | 2·82 | 4·26 | 6·05 | 1·98 | 4·25 | 2·82 | 4·80 | 2·50 | — | — | — |
| 120 | 3·32 | 3·62 | 5·85 | 2·05 | 4·50 | 2·67 | 4·50 | 2·67 | — | — | — |
| 130 | 3·50 | 3·43 | 5·85 | 2·05 | 4·75 | 2·53 | 4·75 | 2·52 | 5·4 | 2·22 | Templ. |
| 140 | 3·53 | 3·40 | 5·78 | 2·07 | 4·75 | 2·53 | 4·85 | 2·47 | — | — | — |
| 150 | 4·16 | 2·89 | 6·55 | 1·83 | 4·50 | 2·67 | 4·75 | 2·52 | — | — | — |
| 160 | 5·12 | 2·34 | 7·03 | 1·71 | 5·50 | 2·18 | 5·20 | 2·31 | — | — | — |
| 170 | 6·30 | 1·91 | 7·85 | 1·53 | 6·00 | 2·00 | 5·50 | 2·18 | 6·16 | 1·95 | Above |
| 180 | 7·25 | 1·66 | 8·60 | 1·39 | 5·50 | 2·18 | 5·60 | 2·14 | 4·55 | 2·64 | Nasal |
| Average angle subtended | — | 3·22 | — | 1·89 | — | 2·65 | — | 2·51 | — | 2·30 | — |
| Snellen acuity | — | 6/9·7 | — | 6/5·7 | — | 6/8 | — | 6/7·5 | — | 6/6·9 | — |
| Mean diff. from mean | — | ·806 | — | ·205 | — | ·244 | — | ·184 | — | ·233 | — |

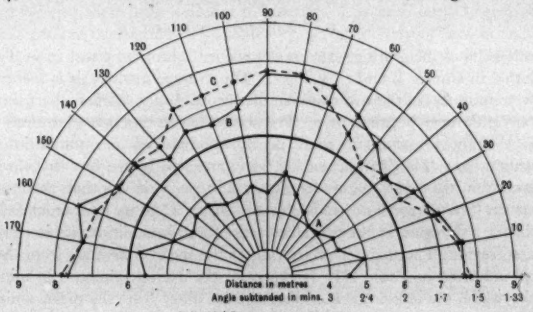
| 1 | 7 | | 8 | | 9 | | 10 | | 11 | | 12 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Observer | H. H. E. | | H. H. E. | | H. H. E. | | H. H. E. | | H. H. E. | | H. H. E. | |
| Lens used | −0·75 D.S. | | −1·00 D sphere | | | | −1·37 D.S. | | −2·00 D.S. | | −3·00 D.S. | |
| | | | 15 mm. from cornea | | 19 mm. from cornea | | | | | | | |
| Diameter of pupil | 7 mm. | | 7 mm. | | 7 mm. | | 7 mm. | | 7 mm. | | 7 mm. | |
| Meridian of eye | *d* | δ | *d* | δ | *d* | δ | *d* | δ | *d* | δ | *d* | δ |
| 10 | 5·40 | 2·22 | — | — | 6·00 | 2·00 | 6·80 | 1·76 | 7·00 | 1·71 | 7·30 | 1·64 |
| 20 | 4·50 | 2·67 | 6·00 | 2·00 | 5·80 | 2·07 | 6·40 | 1·88 | 6·54 | 1·86 | 7·00 | 1·71 |
| 30 | 5·20 | 2·32 | 5·66 | 2·12 | 5·66 | 2·12 | 6·50 | 1·85 | 6·85 | 1·75 | 6·65 | 1·80 |
| 40 | 5·15 | 2·33 | 5·75 | 2·09 | 5·75 | 2·09 | 6·30 | 1·91 | 6·50 | 1·85 | 6·55 | 1·83 |
| 50 | 5·35 | 2·25 | 5·50 | 2·18 | 5·33 | 2·25 | 5·70 | 2·10 | 6·70 | 1·79 | 7·55 | 1·59 |
| 60 | 5·35 | 2·25 | 5·75 | 2·09 | 5·25 | 2·29 | 7·00 | 1·71 | 7·45 | 1·61 | 7·30 | 1·64 |
| 70 | 6·20 | 1·94 | 6·00 | 2·00 | 5·66 | 2·12 | 7·65 | 1·57 | 8·00 | 1·50 | 7·20 | 1·67 |
| 80 | 6·35 | 1·89 | 6·00 | 2·00 | 6·50 | 1·85 | 7·40 | 1·62 | 8·10 | 1·48 | 8·05 | 1·49 |
| 90 | 5·75 | 2·09 | 6·25 | 1·92 | 7·00 | 1·71 | 7·35 | 1·63 | 8·10 | 1·48 | 7·95 | 1·51 |
| 100 | 6·05 | 1·98 | 6·50 | 1·85 | 6·50 | 1·85 | 7·45 | 1·61 | 7·50 | 1·60 | 7·90 | 1·52 |
| 110 | 6·05 | 1·98 | 6·25 | 1·92 | 6·50 | 1·85 | 7·20 | 1·67 | 6·80 | 1·76 | 6·67 | 1·80 |
| 120 | 5·55 | 2·16 | 6·50 | 1·85 | 6·00 | 2·00 | 7·65 | 1·57 | 6·75 | 1·78 | 7·00 | 1·71 |
| 130 | 6·00 | 2·00 | 6·00 | 2·00 | 6·25 | 1·92 | 7·60 | 1·58 | 6·45 | 1·86 | 6·20 | 1·94 |
| 140 | 6·25 | 1·93 | 6·50 | 1·85 | 6·75 | 1·78 | 7·35 | 1·63 | 6·60 | 1·82 | 6·25 | 1·92 |
| 150 | 6·20 | 1·94 | 6·75 | 1·78 | 7·00 | 1·71 | 7·70 | 1·56 | 6·50 | 1·85 | 6·50 | 1·85 |
| 160 | 6·35 | 1·88 | 7·00 | 1·71 | 7·33 | 1·64 | 7·30 | 1·65 | 7·05 | 1·71 | 7·15 | 1·68 |
| 170 | 6·35 | 1·88 | 7·00 | 1·71 | 7·33 | 1·64 | 7·90 | 1·52 | 7·15 | 1·68 | 7·00 | 1·71 |
| 180 | 6·00 | 2·09 | 6·25 | 1·92 | 6·50 | 1·85 | 7·80 | 1·54 | 7·85 | 1·53 | 7·80 | 1·54 |
| Average angle subtended | — | 2·09 | — | 1·94 | — | 1·93 | — | 1·69 | — | 1·70 | — | 1·70 |
| Snellen acuity | — | 6/6·3 | — | 6/5·8 | — | 6/5·8 | — | 6/5 | — | 6/5 | — | 6/5 |
| Mean diff. from mean | — | ·169 | — | ·112 | — | ·167 | — | ·122 | — | ·115 | — | ·113 |

Table IV (*continued*)

| 1 | 13 | | 14 | | 15 | | 16 | | 17 | |
|---|---|---|---|---|---|---|---|---|---|---|
| Observer | H. H. E. | | H. H. E. | | H. H. E. | | H. H. E. | | H. H. E. | |
| Lens used | −1·00 D.S. / −0·37 D.C. axis 100° | | −1·00 D.S. / −0·37 D.C. axis 100° | | −1·00 D.S. / −0·37 D.C. axis 125° | | −1·00 D.S. / −0·37 D.C. axis 170° | | −1·25 D.S. / −0·25 D.C. axis 150° | |
| Diameter of pupil | 5·5 mm. | | 6 mm. | | 5·5 mm. | | 6 mm. | | 7 mm. | |
| Meridian of eye | *d* | *s* | *d* | *s* | *d* | *s* | *d* | *s* | *d* | *s* |
| 10 | 7·20 | 1·67 | 7·00 | 1·71 | 7·67 | 1·57 | 7·33 | 1·63 | 7·50 | 1·60 |
| 20 | 6·33 | 1·90 | 6·00 | 2·00 | 7·20 | 1·67 | 6·67 | 1·80 | 7·05 | 1·70 |
| 30 | 5·90 | 2·03 | 5·90 | 2·03 | 6·80 | 1·77 | 6·25 | 1·92 | 6·40 | 1·88 |
| 40 | 6·50 | 1·85 | 6·50 | 1·85 | 6·30 | 1·90 | 6·33 | 1·90 | 6·45 | 1·86 |
| 50 | 6·10 | 1·97 | 7·00 | 1·71 | 6·80 | 1·77 | 6·50 | 1·85 | 6·20 | 1·94 |
| 60 | 6·50 | 1·85 | 6·75 | 1·78 | 7·20 | 1·67 | 6·80 | 1·77 | 6·70 | 1·79 |
| 70 | 7·20 | 1·67 | 7·30 | 1·64 | 7·67 | 1·57 | 6·50 | 1·85 | 7·15 | 1·68 |
| 80 | 7·20 | 1·67 | 7·33 | 1·63 | 7·67 | 1·57 | 7·00 | 1·71 | 7·70 | 1·56 |
| 90 | 6·75 | 1·78 | 6·90 | 1·74 | 7·67 | 1·57 | 7·33 | 1·63 | 7·75 | 1·55 |
| 100 | 6·25 | 1·92 | 6·33 | 1·89 | 6·80 | 1·77 | 7·50 | 1·60 | 7·50 | 1·60 |
| 110 | 6·20 | 1·94 | 5·90 | 2·03 | 6·60 | 1·82 | 7·67 | 1·57 | 7·35 | 1·63 |
| 120 | 6·20 | 1·94 | 6·67 | 1·80 | 6·70 | 1·79 | 6·50 | 1·85 | 7·35 | 1·63 |
| 130 | 5·75 | 2·09 | 5·90 | 2·03 | 6·30 | 1·90 | 6·67 | 1·80 | 6·75 | 1·78 |
| 140 | 6·00 | 2·00 | 5·50 | 2·18 | 6·80 | 1·77 | 6·80 | 1·77 | 6·90 | 1·74 |
| 150 | 6·25 | 1·92 | 7·33 | 1·63 | 6·90 | 1·74 | 6·50 | 1·86 | 6·85 | 1·75 |
| 160 | 7·25 | 1·65 | 7·67 | 1·57 | 7·67 | 1·57 | 6·50 | 1·85 | 7·20 | 1·67 |
| 170 | 7·65 | 1·57 | 7·67 | 1·57 | 7·30 | 1·64 | 7·33 | 1·63 | 7·30 | 1·64 |
| 180 | 7·67 | 1·57 | 7·33 | 1·63 | 7·67 | 1·57 | 7·33 | 1·63 | 7·80 | 1·54 |
| Average angle subtended | — | 1·83 | — | 1·80 | — | 1·70 | — | 1·75 | — | 1·70 |
| Snellen acuity | — | 6/5·5 | — | 6/5·4 | — | 6/5 | — | 6/5·2 | — | 6/5 |
| Mean diff. | — | ·139 | — | ·156 | — | ·102 | — | ·101 | — | ·097 |

### (2) *Irregular Acuity and Effect of Lenses*

The mean distance figures of Columns 4 and 5 for the unaided eye are plotted in Fig. 4, Curve A. It is seen at once that there is a certain, though not excessive,




A Unaided Eye
B with −1·00 D.S./−0·37 D.C. ax. 125°
C " −1·25 D.S./−0·25 D.C. ax. 150°

Fig. 4. Author's Left Eye

amount of irregularity, the worst meridians being in the neighbourhood of 45° and over the region 110° to 140°; of these two regions the former is rather the poorer. The subjective test indicated that a cylinder was required with its axis along 125°, its power meridian occupying therefore the direction 35°; this agrees fairly well with the worst meridian in the grating result. The effect of this cylinder will be to improve the region round about 35°, but there is indicated almost as much call for a cylinder to supply power over the region 110° to 140°. If, in the usual subjective test, astigmatism is suspected, the examiner endeavours to locate its meridian by asking the patient to name which lines are clearer in contrast with others at right angles. I suspect that, frequently, in his anxiety to locate these perpendicular meridians, he forces the patient by suggestion to name *one* direction and proceeds to "equalize" the lines along this direction with those perpendicular to them. This process may leave out of account other directions requiring almost as much attention. It did so to some extent apparently in my own case.

Unfortunately in my case the amounts concerned are small; but readings were taken with the eye supplied first with the subjective finding

$$-1{\cdot}00 \text{ D.S.}/-0{\cdot}37 \text{ D.C. axis } 125°,$$

and secondly with a correction the cylinder portion of which was found by compounding two cylinders, one of $-0{\cdot}37$ D axis 130° and the second of $-0{\cdot}25$ D axis 25°, these directions being taken from Curve A, Fig. 4. The amount $-0{\cdot}25$ D was decided upon for the second component since the acuity in the 110°–140° region is not quite so poor in the 45° neighbourhood. The resultant is

$$-0{\cdot}41 \text{ D.S.}/+0{\cdot}20 \text{ D.C. axis } 60°,$$

which, added to the $-1{\cdot}00$ D spherical portion of the correction, gives when transposed

$$-1{\cdot}21 \text{ D.S.}/-0{\cdot}20 \text{ D.C. axis } 150°,$$

or, say,

$$-1{\cdot}25 \text{ D.S.}/-0{\cdot}25 \text{ D.C. axis } 150°.$$

The results of the grating tests are set out in Columns 15 and 17 of Table IV and plotted in Curves B and C of Fig. 4. The average acuities, it is seen, are the same (1·70 minutes) in the two cases, so that probably the Snellen chart would not reveal any difference between them. The second correction does, however, appear to have a slight advantage in more nearly approaching an equalization of the various meridians. This is indicated by the curves and by the fact that the "mean difference from the mean" figure for Column 17 (·097) is less than that (·102) for Column 15. The correction responsible for Curve C gives just about as good a result in the 45° region as the other correction, and in addition has improved the 110°–140° region. The manner of arriving at the mean difference from the mean figure is indicated in Table III. It expresses the average amount, in minutes of angle, by which the individual meridian acuities differ from the mean acuity. For an eye perfectly regular, in that its acuity is the same in all meridians, this figure would be zero.

The difference between the two cases is, however, small. My left eye has not

a sufficiently marked irregularity to serve as a good case from this point of view. It will be necessary to test with a few more eyes with more definite irregularities.

Columns 13, 14, and 16 of Table IV give the results of tests with the axis of the cylinder placed at 100°, 100°, and 170°. The average acuity is in each case somewhat reduced. It is interesting to note in these three cases that the mean difference from the mean figure is smallest in the case of the 170° direction, which approaches most nearly to the 150° direction of Column 17.

### (3) *Effect of Pupil Diameter*

The effect on grating acuity of varying the pupil diameter was not systematically explored, but Columns 4, 5, and 6 indicate that within the limits 1·75 to 6 mm. diameter the average acuity increases slightly as the pupil is reduced. The results cannot be explained solely on a diffraction basis; the cornea is not spherical and in addition, as the pupil diameter increases beyond 2 mm. or so, the effects due to the peculiar laminated structure of the crystalline lens are felt.

The mean difference figures of Columns 4 and 5, namely 0·244 and 0·184, indicate that with larger pupil the variations between individual meridians are more marked.

An effort was made to explore isolated portions of the pupil (Column 6), but the results it was possible to obtain are too few to analyse.

### (4) *Effect of Accommodation*

The readings tabulated in Columns 11 and 12 were taken in order to ascertain whether there occurred any changes in the acuity of the various meridians when the eye is in the accommodated state. The distance figures of Columns 4, 10, 11 and 12 are plotted for each meridian of the eye in Fig. 5, rectangular co-ordinates being used in this case. Curve B is plotted from the results obtained with the − 1·37 D sphere; there is no cylinder in position. Curves C and D correspond to the eye accommodated about 0·75 and 1·75 D respectively. There is an unmistakable change in the form of the curves as contrasted with Curve B for the unaccommodated eye.

Bearing in mind that lines in one direction only are viewed at one time, the change in shape of the crystalline lens when accommodating appears to have affected the acuity of the eye in different meridians. The 45° and 90° regions are improved and the 110° to 160° region deteriorated. It is conceivable that a redistribution of the lens material may cause this, but further readings must be taken before anything more definite can be stated. The persistence of the high standard of acuity of the horizontal meridian (for vertical lines) is to be noted; preference for vertical lines is known to be general.

### (5) *The Sensitiveness of the Eye to Change in Lens Power*

Although, as stated in Section IV (1), the original enquiry as to the sensitiveness of the eye to small changes in lens power was postponed in order to deal with the irregular acuity effects that presented themselves, the results provide some informa-

tion under this heading. As negative power is supplied to the eye, the average angle resolved is gradually reduced from 2·65 minutes (Column 4) to the apparent limiting value, for my left eye, of 1·69 minutes, according to Table V.

Table V

Author's Left Eye. Ratio between Lens Power and Acuity.

| Power of lens D | Angle resolved min. | Power added D | Reduction in angle min. | Reduction in angle per dioptre min. |
|---|---|---|---|---|
| 0 | 2·65 | — | — | — |
| −0·75 | 2·09 | −0·75 | ·56 | ·75 |
| −1·00 | 1·94 | −0·25 | ·15 | ·60 |
| −1·37 | 1·69 | −0·37 | ·25 | ·67 |
| | | | Mean | ·67 |

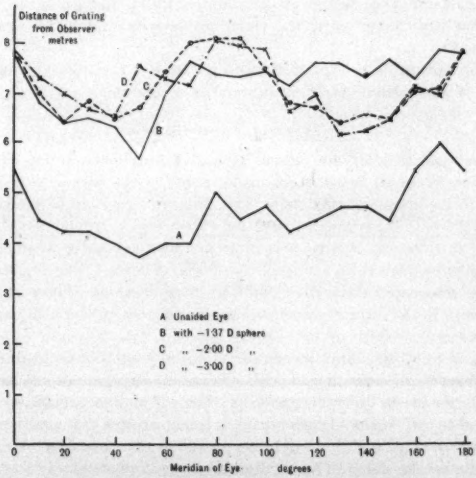



Fig. 5. Author's Left Eye. Effect of Accommodation

The mean reduction in the angle resolved is 0·67 minute per dioptre of lens power; expressed in another way, the "grating badness" increases by about 0·67 minute of angle per dioptre of myopia.

The results obtained by varying the cylinder portion of the correction pointed to the fact that the eye is susceptible also to small variations in cylinder power and

direction (Fig. 4); it appears from these results (Columns 13 to 17 of Table IV) that both the average acuity and the distribution of the acuity over different meridians are affected. Further experiments need to be made, however, before definite figures can be given.

## REFERENCES

(1) W. Swaine. "Paraxial Actions of Ophthalmic Lenses," *Trans. Opt. Soc.* 24 (1922–23) 79.
(2) O. P. Raphael. "Standards of Accuracy for Ophthalmic Prescriptions," *Trans. Opt. Soc.* 24 (1922–23) 75.
(3) Ferree and Rand. *Amer. Journ. Ophth.* Ser. 3 (June, 1925), No. 6.
(4) —— —— *54th Convention of Amer. Ophth. Soc.* (10th July, 1918).
(5) Tscherning. *Physiologic Optics*, 3rd English edn, chapter 10.
(6) E. Jackson. *Practical Aspects of Irregular Astigmatism*, Paper to Section of Ophth. College of Physicians, Philadelphia.

## DISCUSSION

Mr M. Blood: As a microscopist I am quite familiar with the phenomenon of a fine structure that can only be "glimpsed." It is curious that one will sometimes glimpse a structure with difficulty for a second at intervals, and afterwards "hold" it steadily and easily. Illumination is an important factor—both of the object and of the room. The latter should be moderately but not brightly lighted, while the object should be considerably brighter, but the difference must not be too great. It should be interesting to try the effect of varying the illumination of the test box, and perhaps that of the laboratory also.

Fatigue is very important in dealing with border-line resolution, particularly with monocular observation. Personally my eye gets very tired after fifteen minutes' critical monocular work, but binocular vision I can continue much longer. In connexion with this I may say that I feel sure it is a great mistake to try to save strain by using the eyes alternately. It takes so much more effort to see with the "bad" eye that fatigue is increased. Years ago I made the mistake of training my bad eye to make general observations, while keeping the good eye for special work, with the result that, when tested for refraction, my bad eye was 33 per cent. better than normal acuity in spite of a considerable partial cataract, probably traumatic. I bitterly regret this training as the second eye will not "lie down" for long, when using any monocular instrument, but will insist on seeing things.

Another point about difficult resolutions is not to strain or concentrate too much; a sort of casually careful look is best.

I was much interested in Mr Emsley's observation of different astigmatism with different accommodation. When first tested for glasses I was not satisfied with the cylinder correction ordered; experimenting at home I satisfied myself that the axis of astigmatism varied to a certain extent with different accommodation, but my oculist absolutely would not hear of it.

Somewhat analogous is a point that gave me terrible trouble as a rifle shot; when particularly careful and perfectly satisfied that I had got away a good shot I would at times get a wide hit at 4 o'clock, or more rarely at 10 o'clock, on the target. After much investigation I satisfied myself that this was due to my eye having a different optical axis at near and distant focus, e.g. if a rifle in a vice is aimed at a mark when the object is in focus I find the aim is wrong if I focus the front sight. This seems to be a fairly common defect.

It is often useful for a myope rather to under-correct, and a hypermetrope to over-correct, his distant vision, so that he cannot focus the target, but only the front sight.

Mr Emsley: In reply to Mr Blood, I noticed that, during my observations, illumination affected the recognition of the gratings. An appreciable reduction from the 4·8 foot-candles on the gratings sometimes appeared to make them more easily recognized. This phenomenon was not of moment in the main enquiry, however; it may be of sufficient interest to pursue further on another occasion. I have stated the approximate pupil diameter at each series of readings.

The effects caused during the act of accommodation are known in a general way, but need further careful enquiry, together with more work on near tests.

Mr Blood's remarks are interesting and useful as coming from an experienced practical observer.

In reply to Prof. Pollard (see p. 26), work on the eye within the domain of optics is continually skirmishing on the border-lines between this branch of physics and the subjects of physiology and psychology. By students of optics this may be regarded as an added trouble or an added interest: it depends on the point of view. To be consistent and comprehensive, however, account must be taken of the findings of the sister sciences and I am pleased to have called forth from Prof. Pollard a reminder of the paper to which he alludes and to which I shall be happy to refer.

Of course it must not be forgotten that, in spite of repeated attacks, the Snellen chart has held its own up to the present as the most reliable, taking into consideration the variable intelligence of patients as well as other factors.

Prof. Pollard's remarks concern Mr Swaine's paper more than mine so that I will refrain from further reply.

# LAGRANGE'S THEOREM AND STATIONARY FUNCTIONS

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*MS. received 6th October, 1925. Read, 15th October, 1925.*

*ABSTRACT.* Lagrange's theorem for several variables is expressed in a concise notation and used to evaluate a function which is stationary for small changes in each of its variables.

## Introduction

A word of explanation is needed to justify the appearance of a general mathematical paper in a publication devoted to optics. It is widely recognized that the wave theory of light provides a more adequate basis for the deduction of the properties of optical instruments than is afforded by the concept of groups of rays. In some respects the two theories conflict, the conclusions of the ray theory then being definitely in error. Modifications of the ray concept can be introduced to avoid these false conclusions, but this course must be regarded as a temporary expedient, a concession to the strong hold the ray theory has attained in industrial optics. The case for the construction of formulae consistently from the point of view of the wave theory, whether the results are new or old, need not be pleaded.

In the more advanced portions of the theory progress is greatly facilitated by a few special tools which in themselves have no optical implications. The outstanding feature of the wave theory in its application to explain the properties of instruments is the principle that the time taken by light to pass between two points is, to a first approximation, independent of the particular path taken. Thus the evaluation of functions when they are stationary for small changes in the values of their variables is of great importance in the development of a theory of optical instruments on the basis of wave propagation.

The discussion of this problem leads naturally to expansion by means of Lagrange's theorem. The extended form of this theorem for several variables is rarely encountered even in general mathematical treatises, and a word as to its form and proof has therefore been added.

A specially simple case of the formula to be found has been given previously in deriving the laws for the addition of aberrations. The simplicity in that case arose from the axial symmetry of the instruments considered. This not only reduces the number of variables to be considered but also implies the absence of certain terms in the functions to be considered which greatly facilitates the derivation of the formula itself. Many problems arise in which these restrictions are inconvenient, hence the more general investigation which follows.

THE CONDITIONS FOR STATIONARY VALUE

The general problem is the evaluation of the function $f$ of $n$ variables

$$x_1, x_2, x_3, \dots x_n$$

when these are given the special values which satisfy the $n$ conditions

$$\frac{\partial f}{\partial x_h} = 0 \quad (h = 1, 2, 3, \dots n) \qquad \dots\dots(1).$$

Any other variables which may be present in the function $f$ will, for the purpose of the present investigation, be regarded as constants.

The function $f$ will be regarded as consisting of two parts, one of which will be denoted by $v$, the rest being at most of the second order in the $n$ variables. This part serves to give approximate values to the variables in an attempt to satisfy (1). Let it be

$$b_{00} + \Sigma b_{0h} x_h + \tfrac{1}{2}\Sigma b_{hh} x^2{}_h + \Sigma\Sigma b_{hi} x_h x_i.$$

Then the conditions to be imposed are

$$b_{0h} + b_{hi} x_i + \frac{\partial v}{\partial x_h} = 0 \quad (h = 1, 2, \dots n) \qquad \dots\dots(2),$$

where summation of all like terms is implied by the repetition of a letter, in this case the letter $i$. It is convenient to adopt this convention for all terms in which a particular letter appears an even number of times, such letter then being a dummy and replaceable by any other letter which is not already employed in the term: on the other hand a letter appearing an odd number of times does not imply a summation.

An approximate value for $x_h$ is obtained on the assumption that $v$ may be neglected; let it be denoted by $y_h$. Thus the definition of the $y$'s is

$$b_{0h} + b_{hi} y_i = 0 \qquad \dots\dots(3),$$

or

$$\Delta . y_h + b_{0i} B_{hi} = 0 \qquad \dots\dots(4),$$

where $B_{hi}$ is the minor of $b_{hi}$ in the symmetrical determinant

$$\begin{vmatrix} b_{11} & b_{12} & b_{13} & \dots & b_{1n} \\ b_{21} & b_{22} & b_{23} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ b_{n1} & b_{n2} & b_{n3} & \dots & b_{nn} \end{vmatrix} = \Delta \qquad \dots\dots(5).$$

If $b_{0h}$ is eliminated between (2) and (3) the exact equation for finding the $x$'s becomes

$$(x_i - y_i)\, b_{hi} + \frac{\partial v}{\partial x_h} = 0 \qquad \dots\dots(6),$$

which may be resolved into

$$(x_h - y_h)\, \Delta + \frac{\partial v}{\partial x_i} B_{hi} = 0 \qquad \dots\dots(7).$$

This equation is of the type suitable for expansion by means of the generalized Lagrange's theorem. It will be convenient to write $v_h$ for the value of $\dfrac{\partial v}{\partial y_h}$, all

the variables having been changed from $x$'s to $y$'s, and to define $n$ new quantities by the equations

$$h + \frac{1}{\Delta} v_i B_{hi} = 0 \qquad \ldots\ldots(8).$$

The quantity $h$ thus defined is a first correction to $y_h$ as the value of $x_h$. This notation has the advantage of fitting in with the summation convention, so that $hv_{hi}$ or $h_i v_h$ denote quantities which are to be summed for all values of $h$ but not for $i$. It must be remembered that the suffix $_h$ denotes differentiation with respect to $y_h$, and has effectively nothing to do with the quantity $h$ defined in equation (8).

For the purpose of Lagrange's expansion equation (6) shows that the differential coefficients of $f$ are identical with those of $v$ except in the case when two differentiations have been performed. In this exceptional case the relation is

$$f_{hi} = v_{hi} + b_{hi} \qquad \ldots\ldots(9).$$

The terms which arise involving the $b$'s in this way may be grouped and reduced to forms expressible in terms of $v$ only. Thus (8) gives

$$v_h + ib_{hi} = 0 \qquad \ldots\ldots(10)$$

and leads by substitution in (9) to

$$if_{hi} = iv_{hi} - v_h \qquad \ldots\ldots(11).$$

This equation may be used to derive other relations by differentiation.

## Lagrange's theorem

Lagrange's theorem is a formula for the expansion of a function

$$f(x_1, x_2, x_3, \ldots x_n)$$

in a series of ascending powers of $\xi$'s when the $x$'s are defined in terms of two sets of quantities $X_1, X_2, X_3, \ldots X_n$ and $\xi_1, \xi_2, \xi_3, \ldots \xi_n$ by means of $n$ equations

$$x_m = X_m + \xi_m \phi_m (x_1, x_2, x_3, \ldots x_n).$$

$f$ is first expanded in a series of powers of the $\xi$'s by Taylor's theorem, the coefficients being differential coefficients of $f$ with respect to the $\xi$'s which are all replaced by zero after differentiation. The differential coefficients with respect to the $\xi$'s may be replaced by other coefficients in which the variables are the $X$'s. This substitution enables the $\xi$'s to be made zero before, instead of after, differentiation: in other words the $x$'s are simply replaced by $X$'s. The only difficulty in the expansion is due to the complicated character of the coefficients.

Let $D_m$ and $\delta_m$ be used respectively to denote $\frac{d}{dX_m}$ and $\frac{d}{d\xi_m}$. It is easily seen that $\delta_m f = \phi_m D_m f$ whatever $f$ may be. Thus

$$\begin{aligned}\delta_m \delta_n f = \delta_m (\phi_n D_n f) &= \phi_m D_m (\phi_n) . D_n f + \phi_n D_n (\phi_m D_m f) \\ &= \phi_m D_m \phi_n . D_n f + \phi_n D_n \phi_m . D_m f + \phi_m \phi_n D_m D_n f.\end{aligned}$$

Proceeding in this way it may be seen that the general coefficient is obtained by dividing the letters identifying the variables in every possible way, the $D$'s of any

one section operating on the $\phi$'s of the next section, and the $D$'s of the last section on the function $f$ itself. When there are four variables the following groups occur:

| | | | 1234: | | |
|---|---|---|---|---|---|
| | 1:234: | 2:341: | 3:412: | 4:123: | |
| 12:34: | 13:24: | 14:23: | 34:12: | 24:13: | 23:14: |
| | 123:4: | 234:1: | 341:2: | 412:3: | |
| 1:2:34: | 1:3:24: | 1:4:23: | 2:3:41: | 2:4:13: | 2:1:34: |
| 3:4:12: | 3:1:24: | 3:2:41: | 4:1:23: | 4:2:31: | 4:3:12: |
| 1:23:4: | 1:34:2: | 1:42:3: | 2:34:1: | 2:41:3: | 2:13:4: |
| 3:41:2: | 3:12:4: | 3:24:1: | 4:12:3: | 4:23:1: | 4:31:2: |
| 12:3:4: | 13:4:2: | 14:2:3: | 12:4:3: | 13:2:4: | 14:3:2: |
| 23:4:1: | 24:1:3: | 23:1:4: | 24:3:1: | 34:1:2: | 34:2:1: |
| 1:2:3:4: | 2:3:1:4: | 3:1:2:4: | 1:3:2:4: | 2:1:3:4: | 3:2:1:4: |
| 2:3:4:1: | 3:4:2:1: | 4:2:3:1: | 2:4:3:1: | 3:2:4:1: | 4:3:2:1: |
| 3:4:1:2: | 4:1:3:2: | 1:3:4:2: | 3:1:4:2: | 4:3:1:2: | 1:4:3:2: |
| 4:1:2:3: | 1:2:4:3: | 2:4:1:3: | 4:2:1:3: | 1:4:2:3: | 2:1:4:3: |

These 75 groups, involving 125 terms, indicate clearly enough the labour required to apply the theorem in special cases, and the advantage of simplifying the general formula for a particular type of problem rather than leaving this to be done in an expression complicated by the special features of a particular enquiry. In view of the reductions which will be obtained it is of interest to note the relatively large number of groups in which $f$ is only differentiated once—in this example 52 out of 75.

Terms in which differentiation with respect to some of the $\xi$'s is repeated are of the same general form, the $\phi$'s being raised to the corresponding powers and the additional differentiations with respect to the $X$'s applying to the whole group. Thus

$$\delta_m{}^a\delta_n{}^b f = D_m{}^{a-1}D_n{}^{b-1}(\phi_m{}^a D_m\phi_n{}^b D_n f + \phi_n{}^b D_n\phi_m{}^a D_m f + \phi_m{}^a\phi_n{}^b D_m D_n f).$$

This conclusion may be verified by giving the same value to some of the distinct letters in the previous formula. For example, on replacing $m$ by $n$ the formula with two differentiations becomes

$$\delta_n{}^2 f = 2\phi_n D_n\phi_n D_n f + \phi_n{}^2 D_n{}^2 f = D_n(\phi_n{}^2 D_n f).$$

Such considerations lead to the usual expression for Lagrange's theorem, the leading terms of which in the notation used in the first section of this note ($y_m$ and $m$ written for $X_m$ and $\xi_m\phi_m$ respectively) are

$$f + hF_h + \frac{1}{2!}\frac{\partial}{\partial y_h}(h^2F_h) + hiF_{hi} + hi_hF_i + \frac{1}{3!}\frac{\partial^2}{\partial y_h{}^2}(h^3F_h)$$
$$+ \frac{1}{2!}\frac{\partial}{\partial y_h}(h^2iF_{hi} + h^2i_hF_i + 2hih_iF_h) + hijF_{hij} + h_ijF_{hj} + h_{ij}ijF_h + hi_hj_iF_j$$
$$+ \frac{1}{4!}\frac{\partial^3}{\partial y_h{}^3}(h^4F_h) + \ldots \qquad \ldots\ldots(12).$$

The small letter $f$ has been replaced by the capital $F$ to emphasize a distinction in

the conventions applicable in this formula. In the previous expressions in which summation is implied for more than one letter all possible values are to be taken in every case. In this formula, however, the restriction that every distinct letter refers to a separate quantity is to be observed. Thus in a term in which sums are to be taken for $h$, $i$ and $j$, $h$ may receive in turn all $n$ values, $i$ every value but $h$, and $j$ every value but $h$ and $i$. This restriction is inconvenient, and may be removed at once by recalling that the terms in which differentiation with respect to a letter is repeated may be obtained by giving like values to a corresponding number of letters which have been hitherto distinguished. It follows that the terms with repeated differentiation are the very terms required to make the sums complete with every letter receiving in turn every possible value. The expansion may therefore be written in a form in which each letter occurs only once on the line and once subscript, the numerical coefficients being adjusted on account of the repetition of a particular term which is then liable to occur. For example

$$hijf_{hi} = h^2F_{hh} + 2hiF_{hi}.$$

The expansion may now be written in the concise form

$$\begin{aligned} f &+ hf_h + \frac{1}{2!}hijf_{hi} + hi_hf_i \\ &+ \frac{1}{3!}hijf_{hij} + hij_hf_{ij} + \frac{1}{2!}hij_hf_j + hi_hj_if_j \\ &+ \frac{1}{4!}hijkf_{hijk} + \frac{1}{2!}hijk_hf_{ijk} + \frac{1}{2}hij_hk_if_{jk} + \frac{1}{2!}hijk_hf_{ijk} + hi_hjk_if_{jk} + \frac{1}{3!}hijk_{hij}f_k \\ &+ hij_hk_{ij}f_k + \frac{1}{2!}hij_{hi}k_jf_k + hi_hj_ik_jf_k \\ &+ \dots \end{aligned} \qquad \dots\dots(13).$$

THE FORMULA FOR STATIONARY FUNCTIONS

The solution of the stationary function problem may now be obtained by replacing the $f$'s by $v$'s in (13), paying attention to (11) and the relations derivable from it such as

$$i_jf_{hi} = i_jv_{hi} - v_{hj},$$

the suffix of the dummy being transferred to the $v$ in the extra term. It is at once seen that great simplification is possible. The third term of (13) becomes

$$\tfrac{1}{2}(hiv_{hi} - hv_h),$$

so that the first correction is halved. The second term of the second line becomes $hij_hv_{ij} - hj_hv_j$, and the second correction reduces to $\frac{1}{2!}hiv_{hi}$. This term could also be reduced to $hij_hv_{ij} - hiv_{hi}$, illustrating the like character of a number of terms which would be calculated independently when Lagrange's formula is used directly. From the terms of the next order the degenerate terms are

$$-\frac{1}{2}hij_hv_{ij} - \frac{1}{2!}hij_{hi}v_j - hi_hj_iv_j,$$

leading to the expansion to the fourth approximation for the stationary value of $f$, say $f_s$,

$$f_s = f + \tfrac{1}{2} hv_h + \tfrac{1}{2!} hiv_{hi} + \tfrac{1}{3!} hijv_{hij} + \tfrac{1}{2} hij_k v_{ij}.$$

The continuation of the expansion to higher orders is now simple. The groups of the next higher order are formed which have two letters in the final section, and the contracted group formed in accordance with equation (11). The general coefficient of a group is obtained by placing in the denominator the factorial numbers corresponding to the number of elements in each section. Thus the following term has its correct coefficients

$$\frac{h:i:jklm:np:qr:v}{1!\,1!\,4!\,2!\,2!} = [h:i:jklm:np:qr:v] \text{ say,}$$

but on expansion they are modified as only one typical term of each kind is written down. Thus as the result of the operation of the second section on the third the contribution to a typical term is $4j_iklm$.

The relations derivable from (11) by differentiation may be expressed in the form that interchanges of index of the type $h_i\vec{v}_{hj} = h_j v_{hi}$ may be carried out freely. It at once follows that all terms with the same set of differentiations are identical, the first differentiation of $v$ itself being disregarded. Thus the term $hij_k k_{ij} v_k$ may be represented by (1, 2) indicating that one letter $j$ is differentiated once and another $k$ twice. The terms $hijk_{hi}v_{jk}$ and $hijk_h v_{ijk}$ have the same value. In this notation

$$[hijk:v] = (3)/4! \qquad [h:ijk:v] = (1,2) \times 3/1!3!$$

$$[hi:jk:v] = \{(1,2) + (1,1,1)\} \times 2/2!2! \qquad [hij:k:v] = (3)/3!1!$$

$$[h:ij:k:v] = (1,2) \times 2/1!2!1! \qquad [hi:j:k:v] = (1,2)/2!1!1!$$

$$[h:i:jk:v] = (1,1,1) \times 2/1!1!2! \qquad [h:i:j:k:v] = (1,1,1)/1!1!1!1!$$

making a total of $$5\left[\frac{(3)}{4!} + \frac{(1,2)}{2!} + \frac{(1,1,1)}{2!}\right].$$

The degenerate terms of the next order are

$$-(3) \times 2/3!2! - (1,2) \times 6/3!2! \quad \text{from } [hij:kl:v]$$

$$-\{(1,2) + (1,1,1)\}\, 2 \times 2/1!2!2! \quad \text{from } [h:ij:kl:v]$$

$$-(1,2) \times 2/2!1!2! \quad \text{from } [hi:j:kl:v]$$

$$-(1,1,1) \times 2/1!1!1!2! \quad \text{from } [h:i:j:kl:v]$$

making a total of $-4\left[\frac{(3)}{4!} + \frac{(1,2)}{2!} + \frac{(1,1,1)}{2!}\right]$. The net result is thus exactly one-fifth of the terms given by the ordinary Lagrange formula. This result holds

true generally, the numerical divisor being equal to the number of quantities multiplied in the term, $v$ included. The general formula is thus

$$f_s = f + \frac{1}{2} hv_h + \frac{1}{3}\{[hi:v] + [h:i:v]\} + \frac{1}{4}\{[hij:v] + [h:ij:v] + [hi:j:v] + [h:i:j:v]\} + \ldots.$$

In use this will be contracted to

$$\begin{aligned} f_s = f &+ \tfrac{1}{2}(0) \\ &+ \tfrac{1}{2!}(1) \\ &+ \tfrac{1}{3!}(2) + \tfrac{1}{2}(1,1) \\ &+ \tfrac{1}{4!}(3) + \tfrac{1}{2!}(1,2) + \tfrac{1}{2}(1,1,1) \\ &+ \tfrac{1}{5!}(4) + \tfrac{1}{3!}(1,3) + \tfrac{1}{8}(2,2) + (1,1,2) + \tfrac{1}{2}(1,1,1,1) \\ &+ \tfrac{1}{6!}(5) + \tfrac{1}{4!}(1,4) + \tfrac{1}{12}(2,3) + \tfrac{5}{12}(1,1,3) + \tfrac{5}{8}(1,2,2) \\ &\qquad + \tfrac{5}{3}(1,1,1,2) + \tfrac{1}{2}(1,1,1,1,1) \\ &+ \ldots. \end{aligned} \qquad \ldots\ldots(14).$$

The numerical coefficients are those in the expansion of

$$\frac{1}{(n+1)!}\frac{\partial^{n-1}}{\partial x^{n-1}}(y^{n+1}),$$

the number of variables entering into the term being $n$, which exceeds by unity the sum of the numbers in the brackets ( ) in equation (14).

In calculating the first term of the expansion it is convenient to note that

$$f - v = b_{00} - \frac{b_{0h} b_{0i} B_{hi}}{2\Delta}.$$

An alternative method of deriving the expansion is to apply Lagrange's theorem to expand

$$v - \tfrac{1}{2}(x_h - y_h)\frac{\partial v}{\partial x_h},$$

where

$$x_h = y_h + \frac{\partial v}{\partial x_i} C_{hi},$$

and the $C$'s are constants satisfying $C_{hi} = C_{ih}$, the condition which permits

suffixes to be interchanged and the whole expansion to be inferred from the case when there is only one variable. In this case the coefficient of $C^n$ is

$$\frac{1}{n!}\frac{\partial^{n-1}}{\partial x^{n-1}}\left[\left(\frac{\partial v}{\partial x}\right)^n \frac{\partial}{\partial x}\left\{v - \tfrac{1}{2}(x-y)\frac{\partial v}{\partial x}\right\}\right]_{x-y}$$
$$= \frac{1}{n!}\frac{\partial^{n-1}}{\partial y^{n-1}}\left[\frac{1}{2}\left(\frac{\partial v}{\partial y}\right)^{n+1}\right] - \frac{n-1}{n!}\frac{\partial^{n-2}}{\partial y^{n-2}}\left[\left(\frac{\partial v}{\partial y}\right)^n \frac{1}{2}\frac{\partial^2 v}{\partial y^2}\right]$$
$$= \frac{1}{(n+1)!}\frac{\partial^{n-1}}{\partial y^{n-1}}\left(\frac{\partial v}{\partial y}\right)^{n+1}.$$

If now $e$ is a function of the $x$'s and $E$ the same function of the $y$'s, since

$$e = E + (x_h - y_h)E_h + \tfrac{1}{2}(x_h - y_h)(x_i - y_i)E_{hi} + \ldots,$$
$$\frac{\partial e}{\partial x_h} = E_h + (x_i - y_i)E_{hi} + \ldots,$$

it follows that

$$e - \tfrac{1}{2}(x_h - y_h)\frac{\partial e}{\partial x_h} = E$$

when

$$E_h = 0,$$

if the order of $e$ is not greater than two. On writing $f$ for $v + e$, the right side of the expansion agrees with (14), and the left side becomes

$$f - \tfrac{1}{2}(x_h - y_h)\frac{\partial f}{\partial x_h},$$

which is a more general expression for the expansion and reduces to $f$ when the conditions mentioned at first are imposed.

It is proposed to describe a number of applications of this formula in future communications. As an indication of the wide field for which it is suited may be mentioned the construction of Hamilton's Characteristic Function for a system of lenses, the investigation of lenses without axial symmetry, the effect of errors in centering, the calculation of lenses with aspherical surfaces, and many others.

# ON BREWSTER'S BANDS. PART I

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*MS. received, 9th September, 1925. Read, 15th October, 1925.*

*ABSTRACT.* The paper considers the explanation of Brewster's bands and other allied phenomena from the new and very suggestive standpoint proposed by Schuster (*Phil. Mag.* Oct. 1924). When monochromatic light is reflected by or transmitted through two parallel plates in succession, we have a superposition of the Haidinger ring-systems due to the two plates in the sense that the observed intensity in any given direction is the product of the intensities due to either plate separately. Illustrations showing the effect of such superposition in various cases are reproduced with the paper, differential and summational fringe-systems of various orders being observable. When non-homogeneous light is used, the Haidinger rings disappear and along with them also the superposition pattern, leaving only a uniform illumination in the field, except in the special case of the differential system of the first order for two plates of equal thickness. A simple geometrical explanation is thus forthcoming why Brewster's bands can be observed even in non-homogeneous light with thick plates in this case.

## 1. INTRODUCTION

As is well known, coloured interference bands are observed when an extended source of light is viewed in transmission through or by reflection from two glass plates in succession when these are of equal thickness and are held at a suitable small inclination with each other. These bands, first observed by Sir David Brewster, form the basis of the very convenient interference refractometer developed by Jamin, and their explanation is therefore a matter of considerable interest and importance. In a recent and very interesting paper*, Sir Arthur Schuster has proposed a way of regarding the effects observed in this case which is different from that usually given. If we use an extended source of highly monochromatic light, and view it by transmission through or a reflection from a single plate, we should get the well-known Haidinger rings, or interference-curves of constant obliquity representing the variation of the transmitting or reflecting power—as the case may be—of the plate for light of a specific wave-length in different directions. If two plates are used in combination, Schuster suggests that we may regard each of them as producing its own set of Haidinger rings, but that these are superposed in the sense that the observed brightness in any specific direction is determined by multi-

* Schuster, *Phil. Mag.* 48 (Oct. 1924) 609–619.

plying the fractions of the intensity of the original source transmitted—or reflected, as the case may be—by the two plates in succession. For, if we consider a wave-front travelling out from the source in a specific direction, the two plates act on it in succession, and to find their resultant effect, we have only to multiply their transmitting or reflecting powers for the specific angles of incidence concerned. The joint effect of the two plates is thus merely to superpose their interference-patterns in the sense indicated; when the plates are inclined to each other, the centres of the two ring-systems would of course not be coincident. In his paper, Schuster has endeavoured to trace the geometric character of the complex pattern formed by the two non-concentric sets of rings.

While the treatment outlined by Schuster is of undoubted elegance and simplicity, it must be pointed out that his paper does not actually offer an explanation of Brewster's bands. For, as is well known, it is not necessary to use highly monochromatic light to observe these bands, and they may be seen even with white light and thick plates. Under these conditions, the Haidinger rings due to a parallel plate cannot be observed at all, and it is by no means obvious without explanation why, when each plate separately gives merely uniform illumination, their superposition should give any observable fluctuations of intensity in the field. To furnish an explanation of this and to extend the theory from the new standpoint in other directions is the purpose of the present paper. In Part I we shall consider the geometric character of the patterns formed by the superposition of non-concentric ring-systems and explain why Brewster's bands can be observed in non-homogeneous light provided the plates are of equal thickness. In Part II we shall consider the explanation of the general (elliptic) form of Brewster's bands observed in monochromatic light with plates inclined at any angle. In Part III the character of Brewster's bands as observed with plates of doubly refracting crystals will be discussed.

## 2. The Geometric Character of the Superposed Patterns

To illustrate the effect of multiplying the distribution of light-intensity in two sets of rings when their centres are not coincident, we have prepared the prints reproduced in Figs. 1 to 6. Of these, Figs. 1 to 4 represent the superposition of two *identical* ring-systems with gradually increasing distance between their centres. Figs. 5 and 6 represent the effect of superposing two *dissimilar* ring-systems, the centres of which are moderately removed from each other in Fig. 5, and still farther apart in Fig. 6. The figures were prepared by the very simple expedient of printing twice on bromide paper from negatives on which alternate bright and dark rings were spaced in the manner of a Fresnel zone-plate. The same negative was used twice with a suitable displacement for preparing Figs. 1, 2, 3, and 4. For preparing Figs. 5 and 6, two different negatives with ring-systems of unequal size were successively printed on the same sheet of bromide paper.

The law of spacing of the Haidinger rings in a parallel plate of refractive index $\mu$ and thickness $t$ is $2\mu t \cos r = n\lambda$, where $r$ is the angle of refraction within the

plate. If $i$ be the angle of emergence from the plate, the law may be written in the form

$$2t\sqrt{\mu^2 - \sin^2 i} = n\lambda.$$

For moderately small values of $i$, the law of spacing of the rings is very similar to that on a Fresnel zone-plate. Hence Figs. 1 to 4 may be regarded as correctly representing the effect of superposing the Haidinger ring-systems of two plates of *equal* thickness moderately inclined to each other, while Figs. 5 and 6 may be regarded as doing the same for two plates of *unequal* thickness.

The characteristic feature shown in Figs. 1 to 3 is the system of rectilinear equidistant bands running transverse to the line joining the centres of the ring-systems. These bands are very broad in Fig. 1, corresponding to a small inclination of the plates, and become closer and closer together with increasing inclination of the plates until, as in Fig. 4, they are so close together as to be lost to the eye in the general structure of the field. At the same time, other features appear in the superposition pattern, viz. the two external circular fringe-systems shown in Fig. 3, and the internal circular fringe-system shown in Fig. 4. The significance of these other features we shall consider presently. A noteworthy feature of the rectilinear bands is that the central bright one passing through the point midway between the centres of the two systems is always fixed in position and orientation.

We have now to consider the manner in which the superposition patterns are formed. Taking first the case in which the two sets of rings are identical in size, we may, following Schuster, take the centre of each system to be a point of maximum intensity; the radii of the rings having maximum intensity are (approximately) in the ratio of the square roots of successive integral numbers. Referred to a system of co-ordinates in which the line joining the two centres is the axis of $x$ and the origin is a point midway between the centres, the rings are represented by the equations

$$y^2 + (x-a)^2 = R^2 m, \quad y^2 + (x+a)^2 = R^2 n,$$

where $m$ and $n$ are integers and $R$ is a linear constant. From these equations, we readily obtain by addition and subtraction

$$4ax = R^2(n-m) \qquad \text{.....(1)},$$

$$y^2 + x^2 + a^2 = \tfrac{1}{2}R^2(m+n) \qquad \text{.....(2)},$$

$$y^2 + x^2 + 6ax + a^2 = R^2(2n-m) \qquad \text{.....(3)},$$

$$y^2 + x^2 - 6ax + a^2 = R^2(2m-n) \qquad \text{.....(4)},$$

and so on. Equation (1) represents a set of straight lines parallel to the $x$-axis. Equation (2) represents a set of circles with their centres at the mid-point between the centres of the ring-systems. Equations (3) and (4) represent sets of circles with their centres at $x = \pm 3a$, and so on. These loci contain points where the maxima of intensity of one set of rings coincide with the maxima of the other set and the minima of one set with the minima of the other. Alternating between these, we have other loci of similar shape where the maxima of one set coincide with the minima of the other set, and vice versa. Since the intensity in the superposed

patterns is the product of the intensity in the patterns due to the two plates separately, the loci mentioned represent lines along which there is alternately a general strengthening and weakening of the ring-pattern. This is the feature exhibited in Figs. 1 to 4. The systems represented by equations (2), (3), and (4) become prominent in the superposition pattern only when the rectilinear fringes given by (1) are so narrow as not to obscure the structure of the field.

The case of the superposition of two ring-systems of unequal size may be similarly discussed. We have as the equations of the ring-systems

$$y^2 + (x - a)^2 = R_1^2 m, \quad y^2 + (x + a)^2 = R_2^2 n,$$

where $R_1$ and $R_2$ are now different linear constants. Adding and subtracting we have

$$(R_1^2 - R_2^2)(x^2 + y^2 + a^2) + 2(R_1^2 + R_2^2)ax = R_1^2 R_2^2 (n - m) \quad \ldots\ldots(5),$$

$$(R_1^2 + R_2^2)(x^2 + y^2 + a^2) + 2(R_1^2 - R_2^2)ax = R_1^2 R_2^2 (m + n) \quad \ldots\ldots(6),$$

$$(2R_1^2 - R_2^2)(x^2 + y^2 + a^2) + 2(2R_1^2 + R_2^2)ax = R_1^2 R_2^2 (2n - m) \quad \ldots\ldots(7),$$

$$(2R_2^2 - R_1^2)(x^2 + y^2 + a^2) - 2(2R_2^2 + R_1^2)ax = R_1^2 R_2^2 (2m - n) \quad \ldots\ldots(8),$$

and so on. These equations also represent sets of concentric circles, the centres lying on the $x$-axis.

## 3. Phenomena in Non-Homogeneous Light

We are now in a position to discuss the effect of using non-homogeneous light on the appearance of the superposed patterns. An alteration of wave-length changes the linear constant of the Haidinger ring-systems, which consequently expand or contract, as the case may be. If we were considering only each plate separately, the fluctuations of intensity in the field would be wiped out completely, and we would get only uniform illumination. With two plates, as the wave-length is altered, a radial movement of each ring-system occurs and the superposition patterns also alter. The *circular* fringe-systems formed by superposition, whose equations are given in (2), (3), (4), (5), (6), (7), and (8), move radially from their respective centres, and when any appreciable range of wave-lengths is used, they are completely wiped out and disappear. But the rectilinear fringes given in equation (1), which correspond to the differential-pattern for two plates of *equal thickness*, behave otherwise. For the central bright fringe of this system cuts the circular rings orthogonally and therefore remains fixed in position, and the same is also true *approximately* of the other fringes running parallel to it on either side. From equation (1) it can be seen that an alteration of $R$ simply means a change of the width of the rectilinear fringes. Thus in white light the system would remain visible, the central fringes being achromatic and the outer ones coloured. It is only in highly monochromatic light that we can expect to see the circular fringe-systems corresponding to the summationals and differentials of higher order. As a matter of fact, using two plane-parallel glass plates 1 mm. thick and sodium light, the Haidinger rings and also the superposition pattern consisting of the rectilinear fringes corresponding to the differential system, and the circular fringes corresponding to the sum-

Fig. 1

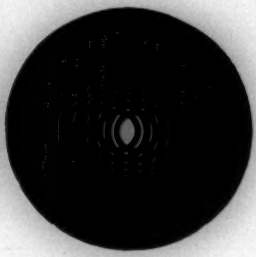

Fig. 2

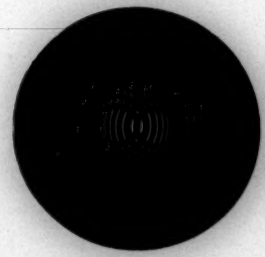

Fig. 3

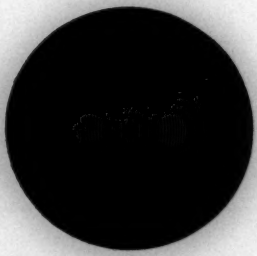

Fig. 4

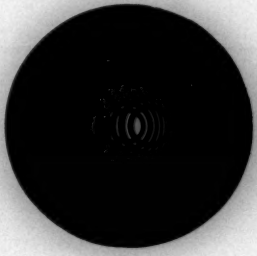

Fig. 5

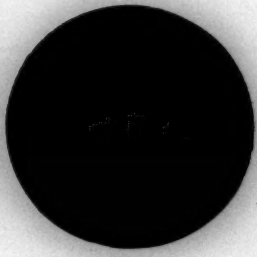

Fig. 6

mationals and the differentials of higher order, can be readily observed. We have studied these features and will describe them more fully in Part II.

The first differential system for two plates of equal thickness appears in equation (1) as a system of straight lines. This is only the case when the Haidinger rings are taken to be spaced exactly according to the law of the square roots of the natural numbers. Actually, as is well known, the Haidinger rings in a glass plate are spaced somewhat differently, the successive rings becoming closer and closer together only up to about an angle of emergence of 45° and then widening out again. As a consequence of this, the differential pattern is not really a system of straight fringes throughout the field, and actually consists of a system of closed ellipses. The modification of the theory necessary on this account and the special phenomena observed in crystalline plates will be considered more fully in Parts II and III of the paper.

We have to thank Mr Asutosh Dey for preparing the illustrations appearing with this paper.

# REVIEWS

*Helmholtz's Treatise on Physiological Optics*, translated from the third German edition. Edited by Professor JAMES P. C. SOUTHALL. Vol. II. *The Sensations of Vision*. Pp. viii + 480; 84 Figs., 3 Plates. (The Optical Society of America, 1924.) Price $7 per volume ($21 for the set of three volumes)*

This, the second volume of Professor Southall's translation of Helmholtz's *Handbuch der Physiologischen Optik*, deals with the sensations of vision. In addition to the matter by Helmholtz himself there are various notes interposed by W. Nagel and von Kries and extensive appendices by both of these authors and by Mrs Christine Ladd-Franklin. Some of the contributions by v. Kries and that by Mrs Ladd-Franklin were written specially for this edition.

The complete volume runs to about 480 pages, of which the appendices occupy 156. There are 10 pages of bibliography, covering the period (1911–1924) since the publication of the third German edition. This is not claimed to be complete, but we have not noticed any serious omissions.

The book is well printed in readable type. There are rather more misprints than one usually expects in a production of this kind, but they are all of a minor nature. The illustrations, which are mostly of a simple diagrammatic character, are well reproduced.

In the double capacity of translator and editor Professor Southall is to be congratulated. As regards the translation we have no criticism at all to offer. The text runs so naturally and fluently that it is difficult to realize that it is a translation.

As regards the editing, one is tempted to let a work so efficiently accomplished pass without criticism, but truth demands that we should point out certain defects.

In some of the editorial footnotes we find statements quoted which are either incomplete or incorrect and which could not have been included had the editor given much personal thought to their implications. For example, in a footnote to p. 33 it is stated that a bright line on a dark background must be at least 3·5 seconds in angular width to be perceived. This, of course, is incorrect. The recognition of bright points or lines against dark backgrounds is purely a question of adequate light. The finest slit or point is visible provided more than the threshold amount of light reaches the eye from it—witness the visibility of stars, or of microscopically thin fibres when strongly illuminated.

Another respect in which we feel the editor's judgment has been defective is in the inclusion of the Appendix by Mrs Ladd-Franklin. We do not personally agree that it is desirable to burden each fresh edition of a classical work of this character with a new appendix by some modern writer. If this practice goes on as it has begun it will be difficult for future generations who may wish to consult the original work of Helmholtz to discover this among the mass of appendices in which it may then be concealed.

If, however, we waive this objection to new matter, the editor is to be congratulated on having the valuable appendices of v. Kries brought up to date, but, for reasons which will appear later, we cannot congratulate him on Mrs Ladd-Franklin's article.

To come to the subject-matter of the book, anything in the nature of a review, as ordinarily understood, of the writings of Helmholtz would be out of place. The book

* This work can be ordered by writing to the Secretary of the Optical Society of America, Rockefeller Hall, Ithaca, N.Y., U.S.A., or may be obtained from the Editor, *The Optician*, 72–78, Fleet Street, London, E.C. 4.

is not a new work requiring vindication or criticism: it is a well-known treatise, of which the nature and importance are fairly generally understood. We shall confine ourselves therefore to a few random observations on the main body of the work.

There is a fair amount of space taken up by matter which would nowadays be regarded as very elementary. For instance, about 30 pages are devoted to the properties of a prism for images at a finite distance! As this is quite irrelevant to the subject-matter of the book it would be within the legitimate province of the editor to delete it from subsequent editions, merely stating, in a few paragraphs, the nature of the problem dealt with by Helmholtz and the lines on which he treated it.

Other occasions when Helmholtz is unduly tedious and elementary are when making clear the distinction between mixing coloured lights and mixing pigments (pp. 122 *et seq.*) and when proving the "centre of gravity" construction for mixed colours on Newton's diagram (pp. 134–8).

The outstanding interest of the present section of the *Handbuch* is, of course, in relation to colour vision. It contains what must still be regarded as the most authoritative collection of data on the varied and complex phenomena of after images and contrast, whether we accept or reject the author's theory of the phenomena. The failure of Helmholtz to give a physiological explanation of simultaneous contrast has been regarded by many critics as a crucial weakness in his theory of the visual process. It is all the more interesting to find in a note by v. Kries (p. 300) that recent work on this subject appears to lead to conclusions differing very little, if at all, from those of Helmholtz.

We find in this volume, of course, the development of the so-called "Trichromatic Theory" of colour vision. It is worth noting in passing that Helmholtz himself gives complete credit for the origination of this theory to Thomas Young. It is introduced on p. 143 as "...the way Young regarded the matter" and is referred to simply as "Young's theory" or "Young's hypothesis" in subsequent pages. Nowhere does Helmholtz make any statement which could imply a claim to even part responsibility for the original conception. His part consisted in covering the skeleton of the theory with a body of explained fact. This he did so well that posterity can hardly begrudge him the bracketing of his name with that of Young which is nowadays customary in describing this theory.

The greatest boon which the present English translation will confer is in enabling many who could only peruse Helmholtz with difficulty, or who had to rely on second-hand information concerning his work, to become familiar with it. Much confusion concerning the trichromatic theory, not only as to its origin but as to what it is and what are its limitations, will be dispelled if the volume is as widely read as it ought to be.

Before leaving Helmholtz and passing on to the appendices, we must note minor points which might cause difficulty to the reader. On p. 129 a statement is made that when equally saturated colours of equal luminosity are mixed the resulting colour is about midway between the components. Helmholtz's meaning here is not too clear: in any case the statement is not recognizable as referring to any quantitative relationship known to present-day colorimetrists.

On p. 146, last paragraph, Seebeck's second division of colour-blindness is identified with Daltonism, or red-blindness, but in the footnote to p. 147 it is referred to as green-blindness. The last paragraph on p. 214, which ends on p. 215, is not very clearly written but it seems obvious that the word "less" occurring in the first line of p. 215 should be "greater."

The appendix by W. Nagel consists of 81 pages devoted to the phenomena of adaptation, twilight vision, and the "Duplicity" theory. In this we have the separate function of the rods and cones considered, and their bearing on such phenomena as dark adaptation, the Purkinje effect, peripheral colour-blindness, etc. discussed.

The first of the appendices by v. Kries treats of the laws of colour mixture in the cases of normal and anomalous colour systems, and the second deals with colour theories, and

discusses, among other matters, the extent to which the results of the Duplicity theory involve modification of the Young-Helmholtz conceptions.

These appendices are valuable statements and contain much important information. They suffer, however, from a very curious defect, particularly evident in those by v. Kries, which, having been brought up to date, may fairly be judged by modern standards. While the writers are clearly masters of the qualitative aspects of the subject, their purview does not include, apparently, anything of the nature of precise experimental work on the various properties of vision such as has been carried out for many years past by compatriots of the translator. Much of this work is very relevant to some of the theoretical questions at issue, yet we find nothing in this book (except, of course, references inserted by the editor) to indicate that the quantitative side of visual experimentation has got beyond what it was in Helmholtz's own time.

Thus, v. Kries, on p. 416, in a note written for the 1924 edition, deals with heterochromatic photometry as though completely ignorant of the work of the last fifteen years. No reference to any of the extensive researches by the numerous American workers who have investigated the problems of heterochromatic photometry, and who have applied the results to the precise measurement of important visual properties, appears in the text. Crude figures by Polimanti for the prismatic spectrum of gaslight are given as illustrating the flicker principle, and the whole treatment of brightness comparison is of a similar elementary and out-of-date kind.

Pulfrich's recent stereo-photometric process is discussed, and although v. Kries realizes that it is not quite good enough to short-circuit all questions of the significance of such measurements by "defining" equality of luminosity in terms of them, as Pulfrich appears to consider possible, his discussion of the questions involved is of a very hypothetical character. The necessary basis for a fairly complete statement of the relationships between "stereo equality" and equality of luminosity is, of course, available in the various papers by H. E. Ives on visual diffusivity and its influence on flicker and other phenomena.

We have already questioned the wisdom of adding the last appendix—that by Mrs Ladd-Franklin. We see no justification for giving to any one of the rival modern theories of colour vision the very considerable and gratuitous advertisement involved in publishing it as part of an edition of Helmholtz's *Handbuch*. As regards the article itself, we must say quite frankly that the literary style of the author tends to provoke an antipathetic attitude on the part of the reader. The supercilious self-complacency with which she insults the intelligence of physicists in general and of several well-known men in particular, is entirely out of place in scientific literature. We are asked to credit some of the best known present-day writers on colour with pre-Newtonian conceptions of coloured light rays; with a complete inability to distinguish between stimulus and sensation, and so on. If we are to believe Mrs Ladd-Franklin, only self-styled psychologists know how to use language without getting into hopeless confusion. Curiously enough, in spite of this, we find more examples of noticeable confusion of diction in this appendix than in the remainder of the volume.

A critical discussion of the Ladd-Franklin theory would call for more space than is available in a review. We shall confine ourselves to drawing attention to several points in which the physics of the author is at fault.

There is the statement that all the colours of the spectrum can be matched by physical mixtures of red, green, and blue lights (p. 459). This assertion, that three physical stimuli are adequate to match all the colours of the spectrum, occurs, differently worded, in three other places (pp. 456, 461, and 467), and is therefore not simply a case of carelessness such as a mere physicist might be guilty of. The statement is, of course, wrong, and to describe the "Grundempfindungen" of Koenig as "lights of particular wave-lengths," as is done on p. 460, and also on p. 461 where one of them is described as "a green of about 505$\mu\mu$," is as serious a case of confusion of wave-lengths with sensations—or whatever the Grund-

empfindungen may be—as we have met. Mrs Ladd-Franklin is apparently able to use the colour triangle without noticing its most obvious characteristics. Thus the primaries of her own tetrachromatic theory are defined—not in the body of this appendix but in a footnote to p. 65—by the statement, "These are the sensations which are produced *in their purity* by about the wave-lengths 576, 505, 470 and a colour a little less yellow than red." The italics are the reviewer's. This statement can only mean that all conceivable colour sensations, including those of the spectrum, can be produced by mixing primaries defined as above. We have only to place these primaries on the colour triangle to realize that such a statement is incorrect. The shape of the spectrum locus on the colour triangle tells us at once that no finite number of primaries taken from the spectrum (or its continuation on the chart, as in the case of the red primary) would be adequate to produce all spectrum colours in their full saturation. In any system of primaries—whether of a trichromatic, tetrachromatic, or multichromatic system—some at least must be of such a nature that they can never be stimulated alone by any objective stimulus. Otherwise they cannot be made to fit the facts of colour mixture as exemplified in the curved shape which the spectrum locus has on any colour triangle.

Mrs Ladd-Franklin appears to have missed this simple fact, well known to Helmholtz and, we believe, to most physicists who have interested themselves in colour vision since his time.

There is only one other point to which space permits reference. On p. 461 (lower half) the author discusses what is *seen* by dichromats who differ from normal trichromats in lacking one of the Grundempfindungen—Koenig primaries—and deduces this from the colour triangle. Having stated correctly that certain lines represent, for normal vision, colours which will all look alike to the dichromat, she then makes the quite illegitimate assumption that the actual colour sensation perceived by the dichromat will be that represented on the triangle by the intersection of these lines with a line joining yellow and blue. She overlooks the fact that the colour *triangle* only exists for trichromatic vision. For dichromats the colour chart is reduced to a *line*. In the case of, say, a green-blind dichromat the complete chart is simply the *RB* side of the normal triangle. No other line on the triangle has any meaning for this type of dichromat and deductions such as the author makes from Fig. 81 have no possible significance.

The foregoing examples will serve to show the quality of the facts adduced and the methods employed by Mrs Ladd-Franklin in this presentation of her theory. That there are points in her general theory which demand serious consideration, and which may very probably enter into any acceptable theory of the colour sensations is undeniable, but such points will require examination and elucidation by others than the author of the theory, whose methods of exposition, if this appendix is typical of them, do not carry us very far.

That the foregoing criticisms of the book are set out in detail is only rendered possible by its general excellence. Where faults are plentiful, the reviewer is obliged to resort to general terms of criticism: where they are few, as in this case, they can profitably be treated in detail, while the good points necessarily receive only general commendation, a state of affairs which may mislead the unwary into forming an erroneous impression of the value of the book. The volume before us fills a long-felt want, and fills it in a manner which merits the thanks of all students of physiological optics to the editor and all those who have assisted him in its production.

J. G.

*Die Geschichte der Emil Busch A.-G., Optische Industrie, Rathenow, vom Wirtschaftswissenschaftlichen Standpunkt.* By Dr KARL ALBRECHT. Pp. 152; 11 Figs. (Erfurt: Gebr. Richters Verlagsanstalt, 1925.) Price G.M. 3.

This book was originally written as a "Dissertation" or thesis, which is an essential part of the German University curriculum in proceeding to a degree; to this has been added an extra chapter dealing with a slightly later phase, viz. up to the introduction of the German Gold Mark. The book is written in the form of a study in industrial economics tracing the activity of the firm over about 125 years; but the author has not overlooked its possibilities from the standpoint of propaganda. True to its type, every scrap of available information connected with its early beginnings has been included, although very little of this is of real interest.

Its principal value to the English members of the optical industry lies in the indications given of the state of affairs in the German industry during the war and subsequently. The statistics given are rather meagre but have a distinct value in throwing some very significant sidelights on the depression in our own optical industry.

For example, the firm has a savings bank for its employees and the interest paid is 4 per cent. plus the ordinary dividend; in 1919 there were 925 employees of whom 544 had accounts in this savings bank of an average amount of 1530 marks; the average wage of these employees amounted to 2820 marks yearly and the interest on the average 1530 marks was 213 marks over and above the fixed 4 per cent., so that the firm was paying a dividend of about 18 per cent. in that year. Now the value of the mark was already going downhill at that time, so that the average yearly wage of 2820 marks among the better paid employees (the worst paid would not bank very much, if anything) throws considerable light on the cause of the depression in our own industry.

P. F. E.

*Critical Potentials associated with Excitation of Alkali Spark Spectra.* By F. L. MOHLER. Scientific Papers of the Bureau of Standards, No. 505. Pp. 27; 8 Figs. (Washington: Government Printing Office, 1925.) Price 10 cents.

The investigations of Paschen and Fowler in recent years on the spectra associated with atoms in various stages of ionization have opened up an important field in spectroscopy. Not the least interesting part of this field is concerned with spectra corresponding to ionization which involves the disturbance of a complete ring of electrons, and it is to this class that the alkali spark spectra belong. In the paper before us these spectra are studied from the point of view of the work required to produce them, the apparatus being first described and then the results for each metal given in detail. The critical potentials were determined by the electrical method as well as by spectroscopic observation of the lines excited. It is interesting to note that the work required to remove an electron from the rare gas shell of an alkali atom is practically identical with that required in the case of the corresponding rare gas itself.

The work appears to have been carried out with the thoroughness which we have been led to expect from the group of spectroscopists to which the author belongs.

F. S.

*Penrose's Annual: The Process Year Book and Review of the Graphic Arts*, Vol. XXVIII. Edited by WM. GAMBLE, F.R.P.S., F.O.S. Pp. xvi + 146. (London: Percy Lund, Humphries & Co., Ltd., 1926.) Price 8*s.* net.

The present volume contains the following articles: "A Review of Process Work," "Printing in Germany To-day," "Aquatone," "The Work of the Private Presses—The Eragny Press," "The New Artistic Epoch: Are we Aware of its Existence?" "The Development of Photographic Processes in the Printing and Allied Trades," "Newspaper

Advertising," "My Month in America—Some Men and Some Things," "The Choice of an Ink," "Photography and Advertising," "New Light on Development Technique," "Some Notes on the Production of Advertising Showcards," "Hand-made Papers," "The August-Hunter Photo-Composing Machine," "Colour Photogravure," "Collotype—Suggestions for Increasing Output," "Rotagravure in Colours in the United States," "The Claybourn Process and 'Wet' Colour Printing," "Men and Materials," "Photo-Lithography and its Possibilities," "Photoline—The Possibilities of Photo-Composing," "The Relative Advantages of Wet and Dry Plates for Process Work," "Celluloid Varnish," "The Present Position of Collodion Emulsion," "Desensitising Wet Collodion Plates," "Standardising Collodion," "The Photo. Image in Offset Printing," "Litho. and Offset Transferring," "Line Negatives for Photo-Litho.," "A Practical Point in Photo-Lithography," "The Triumph of Reel-fed Newspaper Offset Lithography," "Printing on Matt Papers," "The Printer's 'Ideas' Man," and "The Colorway: Quick Colour Printing for Topicals." The volume contains numerous illustrations representing process work during the year 1925, in addition to plates illustrating the articles.

## ABSTRACTS OF PATENT SPECIFICATIONS

230724. Czikan, L. STROBOSCOPES.

Intermittent illumination of a flash-tube is effected by an induction coil controlled by a trembler the period of which is adjustable, so that the flashes can be made to synchronize with the movements of the object under observation.

230882. Taylor, W., and others. POLISHING GLASS.

A felt or other rubber is saturated with a mixture such as resin or pitch, beeswax, and Japan wax or carnauba. It is then hot-pressed and allowed to expand, whereby voids are left between the fibres of the fabric.

231407. Casella & Co., Ltd., and another. THEODOLITES, ETC.

In an arrangement for reading simultaneously two points on a divided circle, the scale is formed on the inner face of an upturned slant edge carried by the fixed frame, while the moving member carries a pair of microscopes, the light from each of which is deflected downwardly by a pair of prisms on to another prism which directs it into a radial eyepiece.

231563. Pictet, L., and another. STEREO-PROJECTION.

A single source of light is used in connexion with two mirrors, so as to produce two parallel spaced beams of light, which are passed through elements to produce polarization in planes at right angles. The left and right hand pictures are projected in superposition on the screen, and are viewed through suitable binocular polarizers.

231637. Taylor, H. D. PROJECTION APPARATUS.

A number of curved mirrors is provided to form at the centre of the projection lens an image of the crater of an elbowed arc free from astigmatism and spherical aberration. In one arrangement shown, the light is first reflected somewhat obliquely from a concave reflector upon a convex mirror placed inside the focal region of the former, and from this the light falls upon another concave mirror by which the ultimate correction is achieved.

231848. Optische Anstalt C. P. Goerz Akt.-Ges. POLARIZING APPARATUS.

A lens made of doubly-refracting material is employed as a polarizer. One of the refracted beams is cut out by a shutter, or is rendered divergent by a glass negative lens.

232255*. Bourdeaux, B. V. C. STEREOSCOPY.

Stereoscopically related images are exhibited alternately, either for direct vision, or by projection. A rotating frame is employed comprising two object carriers and a pair of screens arranged to limit the time during which each image is presented to view, or to unmask them when they are in the proper position for registration.

232362. Taylor, A., and another. SURVEYING INSTRUMENTS.

In a tacheometric telescope, an indicator is employed in connexion with an internal focussing-device to indicate the distance of the anallatic point from the centre of the instrument. The focussing lens is operated through a rack and pinion gearing, the controlling head of which works in connexion with a scale and index.

232569*. Zeiss, C. MICROSCOPES.

The objective, which is aplanatic for its front focal point, is moved to focus the object, the eyepiece remaining fixed relatively to the stage. The objective is mounted in a tube which slides upon the eyepiece, which itself constitutes a telescopic system.

232602*. Rantsch, K., and another. COLOUR FILTERS.

Separated colour images are obtained by the use of screens which transmit light of one colour, and reflect light of the complementary colour. The screens may be made of coloured gelatine, and be mounted diagonally between glass prisms, which are put together to form a rectangular block.

232759. Vickers, Ltd., and another. TELESCOPES.

A sighting-telescope includes a gyroscopically controlled prism for maintaining the target stationary in the field of view, consisting of a triangular portion having an apex angle of about 50°, and a roof-prism portion arranged at right angles to the bisector of the apex angle of the triangular portion.

232792. Vickers, Ltd., and another. MICROSCOPES; PROJECTORS.

The apparatus is designed to enable a projected micro-image of a specimen to be compared with a micro-photograph of a standard. A casing is divided into two main compartments, in one of which is located an illuminating system, the light from which passes through an iris diaphragm directly upon the micro-photograph. The same source is also utilized in conjunction with a suitable condensing and reflecting system to throw light through the object under examination, and form a projected image. Rustless steel plates ultimately reflect the images to be compared upon a pair of screens placed side by side, the iris diaphragm serving to adjust the relative intensities.

232793. Vickers, Ltd., and another. MICROSCOPES.

A special design of inverted projection microscope, to which an auxiliary eyepiece tube may be fitted for direct observation, is shown as adapted to the apparatus described in the preceding notice. The tube is of heavy section, and passes through an axial hole in a steel block, the upper face of which serves to support a tripod stage. Focussing is effected by a nut upon the tube of the microscope.

* Specification published before acceptance.

233323*. Molinié, R. J. BINOCULAR INSTRUMENTS.

Binocular instruments employing reflecting prisms are so designed that the rays travel through the instrument in a transverse plane. Arrangements are described in which the objectives are more or less widely separated as compared with the eyepieces, according as the object to be observed is more or less distant.

233338*. Cesko-Moravska-Kolben Akciova Spolecnost. SEARCHLIGHTS.

A searchlight mounting includes elements which provide for a rotation about five axes, of which two at least are at right angles. In the result, the axis of the searchlight is set at right angles to the plane in which lie the searchlight itself and the straight path of the aircraft under observation, as determined by sound or other locators.

233410. Sloan, R. A. MICROSCOPES.

Adjustments in two directions are provided to ensure that, when the revolving nosepiece is turned, the axis of the operative objective shall coincide with the axis of the instrument. For this purpose, positioning screws are provided to locate the separate objectives or to fix the position of the rotor as each lens comes into place.

233746. d'Albe, E. E. F. TELEVISION APPARATUS.

Light from a picture or image to be transmitted is allowed to fall upon one or more selenium cells, interrupting devices of various frequencies being interposed in the light proceeding from each elemental area. The composite current is then transmitted by wireless to the receiving apparatus, which includes a loud speaker and a series of resonators, each working with a tuned reed bearing a mirror, and corresponding in frequency with one of the interrupters in the transmitter. The presence of any frequency in the received wave sets into vibration the corresponding resonator and reed, and a patch of light is shown upon the appropriate spot on the screen.

234474*. Optische Anstalt C. P. Goerz Akt.-Ges. MOUNTS.

Colour filters or auxiliary lenses for attachment to objectives are provided with a mount having an annular spring and claws which after pressing in are allowed to spring out and secure an internal grip upon the objective mount.

234578. Taylor, A., and another. SURVEYING INSTRUMENTS.

A miners' dial is fitted with a telescope of unit magnification, either end of which can be used for sighting. An auxiliary lens, of a focal length equal to its distance from the stationary compass circle, can be brought into position upon the end of the telescope for reading the circle under magnification.

234878. Graves, A. PROJECTION APPARATUS.

The object to be projected may be formed upon a moving film, or upon a skirt which surrounds the lamp, and is rotated by ascending hot air. The light then falls upon a Mangin or other mirror, which projects an image upon the screen.

235450. Lindl, M., and another. PROJECTION APPARATUS.

A water cell of conical shape is closed at one end by a lens, and at the other by a glass plate. Water flows continuously through the cell. An air-separating chamber is provided at the inlet end, the air being carried away by an ejector action of the outflowing water.

* Specification published before acceptance.

236111. Szepessy, J. TACHEOMETERS.

An image of the circle is projected into the telescope by a microscope and prism system so arranged that its optical axis lies in a plane passing through the axis of rotation, and the edge of the last prism forms the dividing line in the field between the images of the scale and the staff. The vertical circle is divided according to the tangents of the angle of elevation.

236299. Vinten, W. C., and another. PROJECTION APPARATUS.

In connexion with an optical system consisting of a horizontal arc backed by a reflector, a concave lens is placed between the source of light and the condenser, thus forming an enclosed space for dissipating the heat of the arc. The shadow of the carbons may be eliminated by focussing it between the lenses.

236338. Zeiss, C. TESTING LENSES.

The prismatic effect of the lens system under test is ascertained by combining it with a suitable auxiliary system to produce an image of a mark, which is observed through another system including a variable prismatic member. This consists of a plano-convex and a plano-concave element, of which the latter can be rotated about the centre of the common curved surface, so as to vary the inclination of the plane faces to each other. A reading of the displacement when the prismatic effect of the system under test is nullified gives its amount.

236553. Konieczny*. PROJECTION APPARATUS.

An arrangement of plane mirrors placed at an angle to each other, and intersecting upon the optical axis, is employed in connexion with a source consisting of a single filament. The multiple images so formed provide a more even illumination of the slide.

R. J. T.

* Specification published before acceptance.

## CORRESPONDENCE

In my written contribution to the discussion of Mr Fincham's paper on "The Changes in the Form of the Crystalline Lens in Accommodation" (vol. 26, pp. 239–269), I made a mistake which I hope you will allow me to correct. On p. 266, immediately above Fig. A, instead of the words "flexibility in the lens capsule," I ought to have written "resistance in the lens fibres." In my old model the steel strip representing the capsule was stiffened in certain parts, and in the recent discussion I referred to this fact as evidence in favour of Mr Fincham's theory. But on re-reading what I wrote in 1898 (*Trans. Ophth. Soc.* vol. 19), I find no suggestion that I then suspected the real lens capsule of being similarly stiffened. The suggestion that I did then make was that the lens fibres might possibly give a resistance similar in kind to that represented in the model. By correcting this mistake I hope to make it clear that the capsule theory so cogently put forward by Mr Fincham is his, not mine.

PRIESTLEY SMITH.